



ΕΛΛΗΝΙΚΟ
ΑΝΟΙΚΤΟ
ΠΑΝΕΠΙΣΤΗΜΙΟ

Solving the Friedmann equations and studying the scaling factor $a(t)$
for "Toy" Universes
Γεώργιος Στραβογιάννης (AM 156730)



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ΠΡΟΓΡΑΜΜΑ ΜΕΤΑΠΤΥΧΙΑΚΩΝ ΣΠΟΥΔΩΝ

ΠΡΟΧΩΡΗΜΕΝΕΣ ΣΠΟΥΔΕΣ ΣΤΗ ΦΥΣΙΚΗ

Διπλωματική Εργασία

Solving the Friedmann equations and studying the scaling factor $a(t)$ for "Toy"
Universes

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Η παρούσα εργασία αποτελεί πνευματική ιδιοκτησία του φοιτητή («συγγραφέας/δημιουργός») που την εκπόνησε. Στο πλαίσιο της πολιτικής ανοικτής πρόσβασης ο συγγραφέας/δημιουργός εκχωρεί στο ΕΑΠ, μη αποκλειστική άδεια χρήσης του δικαιώματος αναπαραγωγής, προσαρμογής, δημόσιου δανεισμού, παρουσίασης στο κοινό και ψηφιακής διάχυσής τους διεθνώς, σε ηλεκτρονική μορφή και σε οποιοδήποτε μέσο, για διδακτικούς και ερευνητικούς σκοπούς, άνευ ανταλλάγματος και για όλο το χρόνο διάρκειας των δικαιωμάτων πνευματικής ιδιοκτησίας. Η ανοικτή πρόσβαση στο πλήρες κείμενο για μελέτη και ανάγνωση δεν σημαίνει καθ' οιονδήποτε τρόπο παραχώρηση δικαιωμάτων διανοητικής ιδιοκτησίας του συγγραφέα/δημιουργού ούτε επιτρέπει την αναπαραγωγή, αναδημοσίευση, αντιγραφή, αποθήκευση, πώληση, εμπορική χρήση, μετάδοση, διανομή, έκδοση, εκτέλεση, «μεταφόρτωση» (downloading), «ανάρτηση» (uploading), μετάφραση, τροποποίηση με οποιονδήποτε τρόπο, τμηματικά ή περιληπτικά της εργασίας, χωρίς τη ρητή προηγούμενη έγγραφη συναίνεση του συγγραφέα/δημιουργού. Ο συγγραφέας/δημιουργός διατηρεί το σύνολο των ηθικών και περιουσιακών του δικαιωμάτων.



Ευχαριστίες

Με την ολοκλήρωση του Μεταπτυχιακού Προγράμματος θα ήθελα να ευχαριστήσω τους καθηγητές μου για την πολύτιμη γνώση που μου προσέφεραν όσον αφορά την Επιστήμη της Φυσικής.

Ειδικότερα θα ήθελα να ευχαριστήσω τον καθηγητή μου και επιβλέποντα στην παρούσα εργασία Δρ Λεάνδρο Περιβολαρόπουλο κυρίως για το πολύτιμο υλικό που μου παρείχε για την εκπόνηση της εργασίας.



Απόσπασμα

Στην εργασία γίνονται αναφορές για τις εξισώσεις Friedmann, οι οποίες διέπουν την επιταχυνόμενη διαστολή του Σύμπαντος. Μεγάλο ρόλο σε αυτό παίζει ο παράγοντας κλίμακας $a(t)$. Το σύμπαν αποτελείται από πολλά συστατικά, καθένα από τα οποία έχει διαφορετική σχέση πίεσης - πυκνότητας, επηρεάζοντας κατ' επέκταση και τον παράγοντα κλίμακας. Αφού αναλύονται τα μεγέθη και οι αποστάσεις, καθώς και φαινόμενα όπως το CMB, οι BAO και τα SNIa, που μας δίνουν πληροφορίες για τη διαστολή του σύμπαντος, στη συνέχεια γίνεται εκτενής ανάλυση στο maximum likelihood estimation για την εύρεση των best fit parameter. Αναφέρονται τα κοσμολογικά μοντέλα που προτείνονται ως αντίβαρο στα αναπάντητα προβλήματα που έχουν προκύψει από το Λ CDM. Εν τέλει, με την χρήση μετρήσεων και της συνάρτησης χ^2 γίνεται σύγκριση των μοντέλων και σχεδιάζεται η εξέλιξη του παράγοντα κλίμακα με το μοντέλο που επιλέχθηκε, ενώ δίνεται μια εικόνα της συσχέτισης των παραμέτρων ανάλογα το μοντέλο και των μετρήσεων που χρησιμοποιούνται.



Abstract

This work discusses the Friedmann equations, which govern the accelerated expansion of the Universe. A crucial role in this process is played by the scale factor $a(t)$. The universe consists of multiple components, each possessing a distinct pressure-density relationship (equation of state), which consequently affects the evolution of the scale factor. After analyzing cosmological quantities and distances, as well as phenomena such as the Cosmic Microwave Background (CMB), Baryon Acoustic Oscillations (BAO), and Type Ia Supernovae (SNIa)—which provide valuable information regarding cosmic expansion—an extensive analysis using maximum likelihood estimation is conducted to determine the best-fit parameters. Furthermore, various cosmological models are presented as alternatives to address the unresolved issues associated with the Λ CDM model. Finally, by utilizing observational data and the χ^2 function, a comparison of these models is performed and the evolution of the scale factor according to the preferred model is designed, providing an overview of the parameter correlations depending on the specific model and the datasets employed.



Contents

Chapter 1

1.1 Introduction.....	1
1.2 FRW metric.....	2
1.2.1 Energy- momentum tensor.....	3
1.3 Friedmann equations.....	5
1.3.1 Critical density and present values.....	5
1.3.2 Dimensionless critical parameters.....	6
1.3.3 Constant equation of state.....	7
1.3.4 Dynamical equation of state.....	7
1.4 “Toy” universes.....	9
1.4.1 Single-component universe.....	9
1.4.2 Two-component universe.....	9
1.4.3 Multi-component component.....	10
1.5 The cosmological redshift z	11
1.6 Hubble’s law.....	13
1.6.1 Definition.....	13
1.6.2 Hubble’s constant.....	14
1.6.3 Physical density parameters.....	14



1.6.4 Derivation.....	15
1.7 Distances in cosmology.....	17
1.7.1 Theoretically defined distances.....	17
1.7.2 Observable distances.....	18
 Chapter 2	
2.1 Hubble's law.....	22
2.2 Dark energy.....	24
2.3 Supernovae types.....	25
2.4 Baryonic Acoustic Oscillation (BAO).....	31
2.5 Drag epoch and sound horizon.....	32
2.6 BAO measurements.....	33
2.7 CMB measurements.....	34
 Chapter 3	
3.1 Bayes Theorem.....	37
3.2 χ^2 function.....	38
3.2.1 Independent measurements.....	38
3.2.2 Non- independent measurements.....	38
3.2.3 "Chi-by-eye".....	40



3.3 Likelihood function.....	40
3.3.1 Fisher matrix.....	40
3.4 Marginalization.....	43
3.5 Confidence limits.....	44
3.5.1 Constant chi-square boundaries as confidence limits.....	44
3.5.2 Errors.....	45
3.6 Monte Carlo.....	46
3.6.1 Monte Carlo Marcov Chain (MCMC) method.....	46
3.6.2 Hamiltonian Monte Carlo.....	47
3.7 AIC/BIC.....	49
3.8 χ^2 for CMB data.....	50
3.9 χ^2 for CMB data.....	50
3.9.1 6dFGS and WiggleZ.....	51
3.9.2 SDSS.....	52
3.9.3 Ly-a.....	53
3.10 χ^2 for SNIa data.....	54
3.11 Total χ^2 function.....	55



Chapter 4

4.1 Spatially-flat ΛCDM.....	57
4.1.1 Theoretical challenges.....	58
4.1.2 Observational challenges.....	60
4.1.3 Maximum Likelihood Estimation.....	62
4.2 Spatially-flat w-CDM model.....	62
4.3 Scalar field dark energy models (Quintessence).....	63
4.3.1 Chevalier-Polarski-Linder (CPL) model.....	66
4.4 Phantom energy model.....	67
4.5 Calculation of χ^2 and contours.....	67
Conclusion.....	75
References.....	76

List of figures

1.1: The spacetime of the universe can be foliated into flat, positively curved or negatively curved spatial hypersurfaces (Baumann, D.).....	3
1.2: Evolution of the energy densities in the universe (Baumann, D.).....	4
1.3: Evolution of the scale factor in Λ CDM universe and in "Toy universes"....	11
1.4: Figure about the diameter angular distance (Baumann, D.).....	18
1.5: Figure about the luminosity distance (Baumann, D.).....	19
2.1: The expansion of the universe. The physical distance between fixed co-moving coordinates increases with time (https://www.sr.bham.ac.uk/~smcgee/ObsCosmo/lectures/ObsCosmo2.pdf)..	23
2.2: Remnants of Supernova N132D, that exploded 3.000 years ago. It is estimated that the progenitor star had a mass of $25 M_{\odot}$ (https://www.sr.bham.ac.uk/~smcgee/ObsCosmo/lectures/ObsCosmo2.pdf)..	25
2.3: Left: the double - degenerate scenario. Right: the single - degenerate mode (Maoz D.,2014).....	27
2.4: Light curves of 10 well observed SN Ia (Riess A. G. et al., 2019).....	30
2.5: The BAO generation description a) early universe b) acoustic waves forming c) Dark Matter perturbation grows d) recombination e) photon decoupling f) Dark Matter and Baryons mix up. (Castander F. J.).....	32
2.6: CMB power spectrum measured by Planck (Tong D., 2019).....	35
4.1: The 1σ - 3σ confidence contours in the parametric space (h, Ω_{0m}) due to the BAO and the BAO + CMB data.....	68
4.2: The 1σ - 3σ confidence contours in the parametric space $(\mathcal{M}, \Omega_{0m})$.The red contour corresponds only to the full Pantheon dataset while the blue contour correspondsto the Pantheon+CMB+BAO data combination.....	68
4.3: Contours in the parametric spaces of all combinations of w CDM parameters.....	70
4.4: The parametric space for the w CDM model with and without Riess point.	72
4.5: The parametric space for all parameters of CPL model with and without Riess point.....	73



4.6: The evolution of scale factor and its error band.....	74
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List of tables

4.1: The best fit - parameters of the Λ CDM model based on different measurements.....	69
4.2: The best fit - parameters for the three models based on measurements from BAO, CMB, Pantheon data.....	69
4.3: The best fit - parameters for the three models based on measurements from BAO, CMB, Pantheon data and Riess point.....	71

Chapter 1

1.1 Introduction

Cosmological Principle

What exists beyond Earth has been an issue since ancient times. Copernicus was the first who pointed out that Earth is not in the center of universe. Later, other cosmologists stated that the Sun is not in the center and that our Galaxy is not in the center and finally, there is no center in the universe. As a result, modern cosmologists have adopted the Cosmological Principle, according to which there is nothing special about our location in the universe. (Ryden B., 2017)

Based on the Cosmological Principle, the universe is considered to be isotropic and homogeneous on large scales (100 Mpc or more). Isotropic means that there are no preferred directions in the universe, while homogeneous means that there are no preferred locations in the universe. (Ryden B., 2017).

Olbers' Paradox

According to the cosmological model of Copernicus, in an infinitely large and infinitely old universe the night sky should have been uniformly bright with starlight. But in reality, the night sky is dark in visible wavelengths with starlight scattered across it. This is called Olbers' Paradox after the astronomer Heinrich Olbers, who wrote a paper on this subject in 1826. The solution to this problem was the theory that the universe has a finite age - and as a result, the light from distant stars hasn't reached us yet - and keeps expanding, something that causes the light from stars to be redshifted to lower photon energies. Thus, nowadays, the cosmological model is the one according to which the universe started from the Big Bang, has finite age and keeps expanding.

1.2 FRW Metric

In general relativity, a fundamental object is the spacetime metric, which turns observer - dependent spacetime coordinates into the invariant line element ds^2 .

Due to the cosmological Principle, space and time are separated because of the symmetry. The four-dimensional line element ds^2 for an expanding universe with a scale factor $a(t)$ can be written in the simplest form: (Baumann D.)

$$ds^2 = dt^2 - a(t)dl^2 \quad (1.1)$$

where dl^2 is the line element of a symmetric 3-space.

The homogeneous and isotropic 3-spaces have constant 3-curvature. There are only 3 options for zero, positive and negative curvature, respectively. Using polar coordinates, the universe is described by the Friedmann - Robertson -Walker metric:

$$ds^2 = dt^2 - a(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right] \quad (1.2)$$

- k represents the curvature of the universe. It can only have three possible discrete values: -1, 0, +1 for negative, zero and positive curvature, respectively, as it is mentioned before. These three types of curvature correspond to open, flat or closed universe, as it is shown in figure 1.1.
- $a(t)$ is the scale factor of the universe. It can be redefined by a constant factor and as a result, the value of the scale factor today can be written as $a(t_0) = 1$
- t is the proper time of a free-falling observer.
- r, θ, φ are the comoving coordinates.

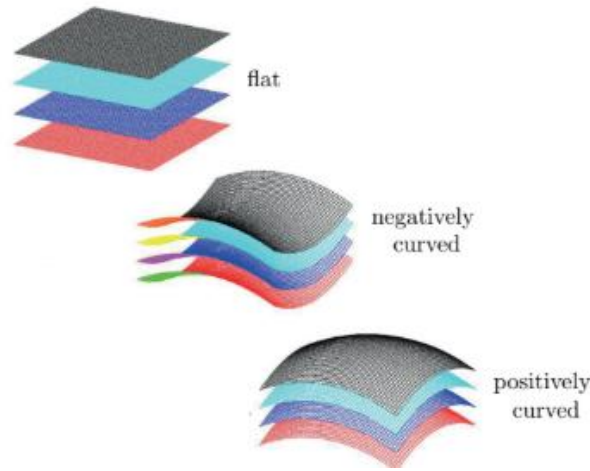


Figure 1.1: The spacetime of the universe can be foliated into flat, positively curved or negatively curved spatial hypersurfaces (Baumann, D.)

The FRW metric can be written in another form:

$$ds^2 = dt^2 - a(t)[d\chi^2 + S^2(\chi)(d\theta^2 + \sin^2\theta d\varphi^2)] \quad (1.3)$$

where

$$S^2(\chi) = \begin{cases} \sin\chi, & k = 1 \\ \chi, & k = 0 \\ \sinh\chi, & k = -1 \end{cases} \quad (1.4)$$

Time t is again the proper time measured by a free - falling observer, while χ is the new radial comoving coordinate.

1.2.1 Energy - momentum tensor

Einstein equation determines the dynamics of the universe:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (1.5)$$

where the Einstein tensor $G_{\mu\nu}$ is a measure of the “spacetime curvature” of the FRW universe, while the energy tensor $T_{\mu\nu}$ is a measure of the “matter content” of the universe.

Because of the homogeneity and isotropy of the universe, it is a requirement that the tensor must be that of a perfect fluid:

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu - P G_{\mu\nu} \quad (1.6)$$

In this equation, ρ and P represent the energy density and the pressure of the fluid, respectively, while U_μ is its four-vector velocity, relative to the observer and as a result, it is equal to (1,0,0,0).

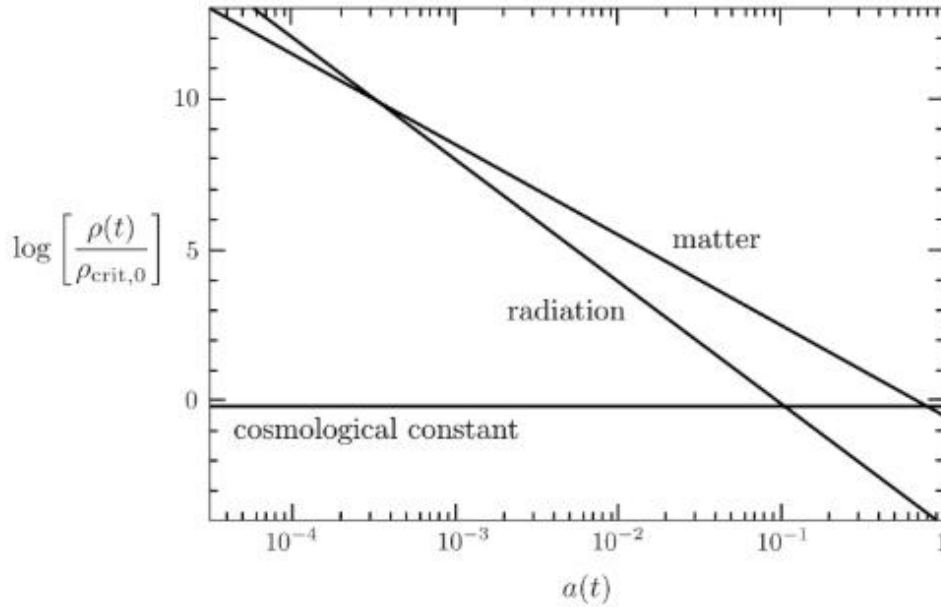


Figure 1.2: Evolution of the energy densities in the Universe. (Baumann D.)

Taking the mixed tensor $T^\mu_\nu = g^{\mu\lambda}T_{\lambda\nu}$ we get:

$$T^\mu_\nu = (\rho + P)U^\mu U_\nu - P\delta^\mu_\nu = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{pmatrix} \quad (1.7)$$

From the covariant conservation equation:

$$\nabla_\mu T^\mu_\nu = 0 \quad (1.8)$$

we derive the continuity equation:

$$\rho + 3 \frac{\dot{a}}{a} (\rho + P) = 0 \quad (1.9)$$

1.3 Friedmann equations

The Einstein tensor is equal to

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad (1.10)$$

where $R_{\mu\nu}$, R are the Ricci tensor and Ricci scalar, respectively.

Calculating the non-zero components of the Einstein tensor we get the Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} \quad (1.11)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3P) \quad (1.12)$$

In these equations, ρ and P represent the sum of contributions to the energy density and pressure in the universe. It should be noted that the subscripts 'r', 'm', 'Λ' in the energy density represent the contribution from radiation, matter and vacuum, respectively. The first Friedmann equation can be written in terms of the Hubble parameter, $H \equiv \frac{\dot{a}}{a}$:

$$H^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} \quad (1.13)$$

1.3.1 Critical density and present values

The critical density for any given Hubble parameter is defined as the energy density ρ_{crit} that would be present if the universe was flat ($k = 0$):

$$\rho_{crit} = \frac{3H^2}{8\pi G} \quad (1.14)$$

To denote quantities evaluated today we use the subscript '0'. For example, the age of the universe today is expressed as $t = t_0$, while

$$\rho_{crit,0} = \frac{3H_0^2}{8\pi G} \quad (1.15)$$

is the critical density today.

1.3.2 Dimensionless critical parameters

Through the use of the critical density, the dimensionless critical parameters are defined as follows:

$$\Omega_{I,0} \equiv \frac{\rho_{I,0}}{\rho_{crit,0}} \quad (1.16)$$

where the subscript 'I' denotes the component (radiation, matter, vacuum). In the literature, the subscript '0' is usually dropped. As a result, Ω_m represents the matter density today in terms of the critical density today.

Cosmic inventory

The universe is filled with a mixture of different matter components. Every source has different contribution to the pressure.

- **Matter**

By using the term "matter" we mean all forms of matter for which the pressure is much smaller than the energy density, $|P| \ll \rho$. In this case, we set $P=0$ in the continuity equation and as a result:

$$\rho \propto a^{-3}$$

- **Radiation**

By using the term "radiation" we mean anything for which the pressure is much smaller than the energy density, $P = \frac{1}{3}\rho$. In this case, from the continuity equation we get:

$$\rho \propto a^{-4}$$

● Dark energy

Recent studies have shown that matter and radiation are not enough for the expansion of the universe, so there must be a mysterious negative pressure component ($P = -p$), which dominates the universe today. In this case the energy density is constant:

$$\rho \propto a^0$$

1.3.3 Constant equation of state

Using the constant equation of state: $w = P/\rho$, we see from the continuity equation that:

$$\rho \propto a^{-3(1+w)} \quad (1.17)$$

To sum up, the solutions for cold dark matter ($w = 0$), radiation ($w = 1/3$), and vacuum energy ($w = -1$) are:

$$\rho \propto a^{-3(1+w)} = \begin{cases} a^{-3}, & \text{matter} \\ a^{-4}, & \text{radiation} \\ a^0, & \text{vacuum} \end{cases} \quad (1.18)$$

1.3.4 Dynamical equation of state

In case of a time dependent equation of state:

$$w(t) = \frac{p(t)}{\rho(t)} \quad (1.19)$$

the continuity equation becomes:

$$\frac{d\rho(t)}{dt} = -3 \frac{\dot{a}}{a} [\rho(t) + p(t)] \quad (1.20)$$

Dividing the equation by $\rho(t)$, the equation takes the form:

$$\frac{\dot{\rho}(t)}{\rho(t)} = -3 \frac{\dot{a}}{a} \left[1 + \frac{p(t)}{\rho(t)} \right] = -3 \frac{1}{a} \frac{da}{dt} [1 + w(t)] \quad (1.21)$$



Thus:

$$\frac{d\rho}{\rho} = -3[1 + w(\alpha)] \frac{d\alpha}{\alpha} \quad (1.22)$$

We integrate this equation as follows:

$$\begin{aligned} \int_{\rho_0}^{\rho} \frac{d\rho'}{\rho'} &= \int_{\alpha_0}^{\alpha} -3[1 + w(\alpha')] \frac{d\alpha'}{\alpha'} \Rightarrow \ln(\rho) - \ln(\rho_0) \\ &= \int_1^a -3[1 + w(\alpha')] \frac{d\alpha'}{\alpha'} \end{aligned} \quad (1.23)$$

Making the substitution:

$$a = \frac{1}{1 + z} \quad (1.24)$$

We get the new differential:

$$da' = d\left(\frac{1}{1 + z}\right) = -(1 + z)^{-2} d(1 + z) = -\frac{dz}{(1 + z)^2} \quad (1.25)$$

Thus:

$$\ln \frac{\rho}{\rho_0} = \int_0^z 3[1 + w(z')] \frac{dz'}{(1 + z')^2} (1 + z') = \int_0^z 3 \frac{[1 + w(z')]}{(1 + z')} dz' \quad (1.26)$$

Finally we obtain:

$$\rho = \rho_0 \exp \left[\int_0^z 3 \frac{[1 + w(z')]}{(1 + z')} dz' \right] \quad (1.27)$$

This equation helps to express the universe expansion since it is not a perfect fluid and its composition changes over time.



1.4 “Toy universes”

1.4.1 Single-component universe

Since each component has different scaling (a^{-3} for matter, a^{-4} for radiation), it is implied that for most of the time the universe was dominated by a single component. For a flat universe with a single component, the Friedmann equation becomes as follows:

$$\frac{\dot{a}}{a} = H_0 \sqrt{\Omega_I} a^{-\frac{3}{2}(1+w_I)} \quad (1.28)$$

Integrating this equation, the scale factor depends on the time as follows:

$$\alpha(t) \propto \begin{cases} t^{\frac{2}{3}}, & \text{matter domination} \\ t^{\frac{1}{3}}, & \text{radiation domination} \\ e^{Ht}, & \Lambda \text{ domination} \end{cases} \quad (1.29)$$

1.4.2 Two-component universe

At some time, radiation and matter densities were equally important. This time was shortly before the cosmic microwave background was released. It is important to denote the scale factor then as:

$$a_{eq} \equiv \frac{\Omega_r}{\Omega_m} \quad (1.30)$$

As a result, at that time (since $\rho_r = \rho_m$ and $\rho_r \propto a^{-4}$, $\rho_m \propto a^{-3}$), the total energy density can be written in the following form:

$$\rho \equiv \rho_r + \rho_m = \frac{\rho_{eq}}{2} \left[\left(\frac{a_{eq}}{a} \right)^4 + \left(\frac{a_{eq}}{a} \right)^3 \right] \quad (1.31)$$

We use the conformal time to solve for the evolution of scale factor:



$$\tau = \frac{dt}{a(t)} \quad (1.32)$$

Thus, the Friedmann equations take the form:

$$(a')^2 = \frac{8\pi G}{3} \rho a^4 \quad (1.33), \quad a'' = \frac{4\pi G}{3} (\rho - 3P) a^3 \quad (1.34)$$

where the primes denote the derivatives with respect to the conformal time.

The solution is written in the form:

$$a(\tau) = a_{eq} \left[\left(\frac{\tau}{\tau_*} \right)^2 + 2 \frac{\tau}{\tau_*} \right] \quad (1.35)$$

where:

$$\tau_* \equiv \left(\frac{\pi G \rho_{eq}}{3} a_{eq}^2 \right)^{-1/2} \quad (1.36)$$

1.4.3 Multi-component universe

For multiple components the general form of the Friedmann equations is:

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \sum_i \rho_i - \frac{k}{a^2} \quad (1.37)$$

$$\frac{\ddot{a}}{a} = - \frac{4\pi G}{3} \sum_i (\rho_i + 3P_i) \quad (1.38)$$

The equations of state are: $p_i = w_i \rho_i$, where w_i is the dimensionless equation of state parameter of the i -th component.

Considering a universe consisted of matter, radiation, curvature and constant Λ , and knowing how the energy density of each component depends on the scale factor, we get:

$$H^2 = H_0^2 [\Omega_{r,0} a^{-4} + \Omega_{m,0} a^{-3} + \Omega_{\Lambda,0} + \Omega_{k,0} a^{-2}] \quad (1.39)$$

where

$$\Omega_{k,0} \equiv -\frac{k}{(a_0 H_0)^2} \quad (1.40)$$

is defined as the curvature density parameter.

At the present time the density parameters satisfy the relation:

$$\Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0} + \Omega_{k,0} = 1 \quad (1.41)$$

We used a python code to examine the evolution of the scale factor for Λ CDM in comparison to "Toy universes", theoretical models in which the universe has one component. The results of the code are in Fig. 1.3:

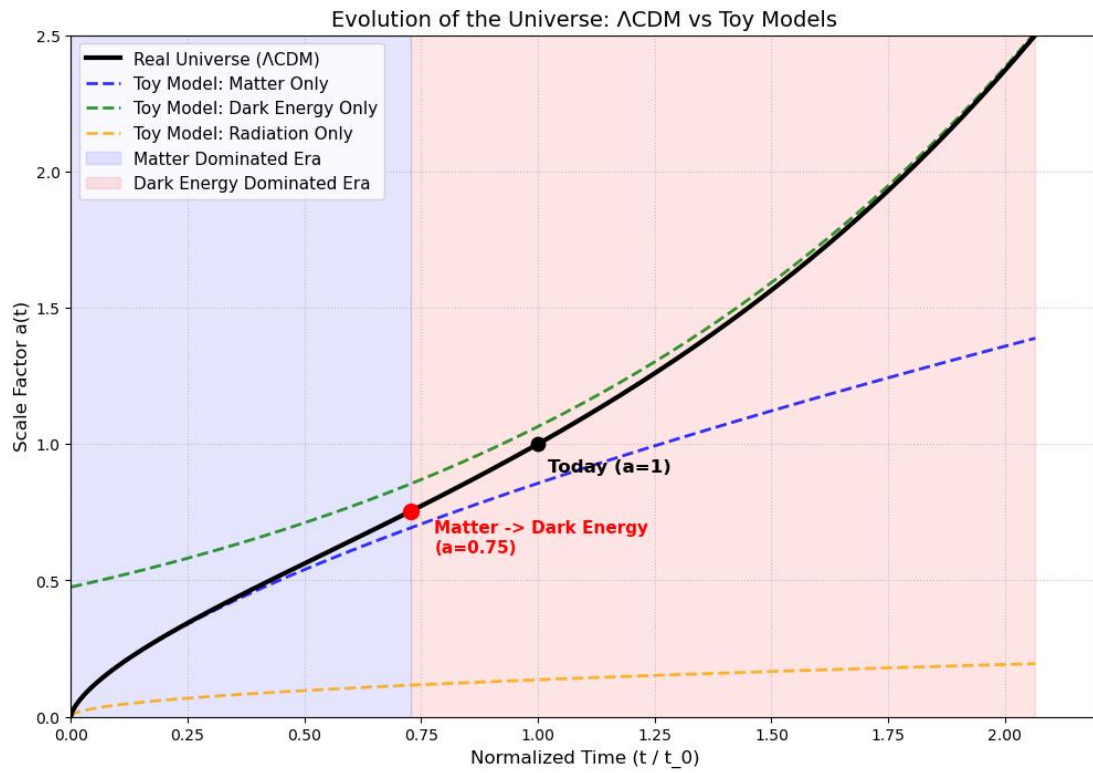


Fig. 1.3: Evolution of the scale factor in Λ CDM universe and in "Toy universes"

From this figure, we see that radiation domination era was very brief, while dark energy started to dominate when a was approximately equal to 0.75.

1.5 The cosmological redshift z

Redshift is a significant phenomenon in cosmology. Everything we know about the universe is inferred from the light emitted by distant objects. To interpret the observations correctly, we need to take into account the finite

speed of light and the expansion of the universe. As a result, the wavelength of the light from distant objects is stretched before reaching us. (Baumann D.). Consider a photon emitted radially from a distant galaxy at time $t = t_E$ and received by us on Earth at time $t = t_R$. Because of that, $ds=d\phi=d\theta=0$. For a flat universe, the FRW metric becomes as follows:

$$c^2 dt^2 = a^2(t) d\chi^2 \Rightarrow \frac{c^2 dt^2}{a^2(t)} = d\chi^2 \quad (1.42)$$

Since the photon is incoming:

$$\frac{cdt}{a(t)} = -d\chi \Rightarrow \int_{t_E}^{t_R} \frac{cdt}{a(t)} = - \int_{x_E}^{x_R} d\chi \quad (1.43)$$

Considering that the photon is observed at Earth, we have $x_R = 0$:

$$\int_{t_E}^{t_R} \frac{cdt}{a(t)} = \int_0^{x_E} d\chi \quad (1.44)$$

Suppose that there is another photon emitted at $t = t_E + \delta t_E$ and received at $t = t_R + \delta t_R$. Then:

$$\int_{t_E + \delta t_E}^{t_R + \delta t_R} \frac{cdt}{a(t)} = \int_0^{x_E} d\chi \quad (1.45)$$

Combining 1. and 1. we get:

$$\int_{t_E + \delta t_E}^{t_R + \delta t_R} \frac{cdt}{a(t)} = \int_{t_E}^{t_R} \frac{cdt}{a(t)} \quad (1.46)$$

We add $\int_{t_E}^{t_E + \delta t_E} \frac{cdt}{a(t)}$ in both sides of the equation so that:

$$\int_{t_E}^{t_R + \delta t_R} \frac{cdt}{a(t)} = \int_{t_E}^{t_R} \frac{cdt}{a(t)} + \int_{t_E}^{t_E + \delta t_E} \frac{cdt}{a(t)} \quad (1.47)$$

As a result:

$$\int_{t_R}^{t_R + \delta t_R} \frac{cdt}{a(t)} = \int_{t_E}^{t_E + \delta t_E} \frac{cdt}{a(t)} \quad (1.48)$$

For $\delta t_R, \delta t_E \ll$, $a(t)$ can be taken as constant in both integrals. Thus:

$$\frac{\delta t_R}{a(t_R)} = \frac{\delta t_E}{a(t_E)} \Rightarrow \frac{\delta t_R}{\delta t_E} = \frac{a(t_R)}{a(t_E)} \quad (1.49)$$

The definition of redshift is as follows:

$$z \equiv \frac{\lambda_R - \lambda_E}{\lambda_E} \Rightarrow 1 + z = \frac{\lambda_R}{\lambda_E} \quad (1.50)$$

Using the relation $\lambda = c\delta t$, we get:

$$1 + z = \frac{\delta t_R}{\delta t_E} = \frac{a(t_R)}{a(t_E)} \quad (1.51)$$

Since we use the subscript '0' for the values of quantities today, the light from a distant object is emitted at time t and observed here at Earth at time t_0 . Also, the scale factor today is set equal to 1. As a result:

$$1 + z = \frac{a(t_0)}{a(t)} = \frac{1}{a(t)} \quad (1.52)$$

1.6 Hubble's Law

1.6.1 Definition

According to the Hubble's law, galaxies appear to recede from us with a recession speed proportional to their distance from us (Hobson M. P. et al., 2006).

$$u = H_0 d \quad (1.53)$$

The constant H_0 is called Hubble constant and measures the current expansion of the universe.

It was named after Edwin Hubble who discovered the expansion of the universe in 1929, when he compared redshifts with distance measurements to nearby galaxies.

1.6.2 Hubble's constant

The constant H is called Hubble's constant and it measures the universe's expansion at any given time t . (Hobson M. P. et al., 2006):

$$H = \frac{\dot{a}(t)}{a(t)} \quad (1.54)$$

As a result, H_0 can be written as:

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} \quad (1.55)$$

Since measurements of the Hubble parameter have shown significant uncertainties, it is convenient to define H_0 as (Baumann D):

$$H_0 \equiv 100 * h \text{ km} * s^{-1} * Mpc^{-1} \quad (1.56)$$

Through this way, h (which is dimensionless) is used to keep track of how uncertainties of H_0 propagate into other cosmological parameters. Today's measurements have been more precise and indicate that:

$$h \sim 0.67 \pm 0.01 \quad (1.57)$$

It is also important the fact that $1/H_0$ gives the Age of the Universe. (Hobson M. P. et al., 2006):

1.6.3 Physical density parameters

Through the use of h we can define also the physical density parameters as [4.]:

$$\omega_{I,0} \equiv \Omega_{I,0} h^2 \quad (1.58)$$

As a result, physical density parameters are independent of the Hubble constant, since from (1.15),(1.16):

$$\Omega_{I,0} \propto \frac{1}{\rho_c} \propto \frac{1}{H_0^2} \quad (1.59)$$

1.6.4 Derivation

It is important to understand how Hubble reached to his Law. [2.,3.]. Consider a photon emitted from a nearby galaxy at time t and received today at time $t = t_0$. Since the galaxy is nearby, $t-t_0 \ll t_0$, so we can expand $a(t)$ around t_0 as:

$$a(t) = a(t_0 - (t_0 - t)) = a(t_0) - (t_0 - t)\dot{a}(t)|_{t=t_0} + \dots \quad (1.60)$$

which gives:

$$a(t) = a(t_0)[1 - (t_0 - t)H_0 + \dots] \quad (1.61)$$

Alongside a photon path it is $ds = d\phi = d\theta = 0$. Considering also a flat universe, the FRW metric (1.2) becomes:

$$\chi = \int_t^{t_0} \frac{cdt'}{a(t')} \quad (1.62)$$

Combining (1.62) with (1.60), we get:

$$\chi = \int_t^{t_0} c a^{-1}(t_0)[1 - (t_0 - t)H_0]^{-1} dt' \quad (1.63)$$

Since $t-t_0 \ll t_0$ and $(a+x)^n \simeq (a+nx)$, we have:

$$\chi \simeq \int_t^{t_0} c a^{-1}(t_0)[1 + (t_0 - t)H_0] dt' \quad (1.64)$$

The evaluation of the integral gives:

$$\chi = ca^{-1}(t_0)[t_0 - t + \frac{(t_0 - t)^2}{2}H_0] \quad (1.65)$$

It is necessary to express $t-t_0$ in terms of the redshift z . It is known that:

$$z = \frac{a(t_0)}{a(t)} - 1 = [1 - (t_0 - t)H_0]^{-1} - 1 \quad (1.66)$$

Since $t-t_0 \ll$,

$$z \simeq [1 + (t_0 - t)H_0] - 1 = (t_0 - t)H_0 \quad (1.67)$$

The inversion gives:

$$(t_0 - t) \simeq \frac{z}{H_0} \quad (1.68)$$

Thus (1.65) becomes:

$$\chi = ca^{-1}(t_0) \left[\frac{z}{H_0} + \frac{\left(\frac{z}{H_0}\right)^2}{2} H_0 \right] = \frac{cz}{a(t_0)H_0} + \frac{cz}{2a(t_0)H_0} \quad (1.69)$$

In first order to z (because $z \ll$):

$$\chi \simeq \frac{cz}{a(t_0)H_0} \Rightarrow a(t_0)H_0\chi \simeq cz \quad (1.70)$$

We can consider z as Doppler shift with recession speed u since $t-t_0 \ll$:

$$u = cz \quad (1.71)$$

The distance from the emitting galaxy due to cosmic expansion at $t = t_0$ is:

$$d = a(t_0)\chi \quad (1.72)$$

Combining (1.70), (1.71), (1.72) we get the Hubble's law:

$$u \simeq H_0 d \text{ for } z \ll \quad (1.73)$$



1.7 Distances in cosmology

For the definition of distances, we consider that we sit at $(t_0, 0, 0, 0)$ while the comoving object that we observe emits light at $(t_e, r, \theta, \varphi)$ and the geometry of spacetime is defined by the line element:

$$ds^2 = dt^2 - a(t)[d\chi^2 + S^2(\chi)(d\theta^2 + \sin^2\theta d\varphi^2)] \quad (1.74)$$

1.7.1 Theoretically defined distances

At first, it is important to define not really observable distances, but that will be useful later in the definition of observable ones (Baumann D.).

The first one is the *metric distance*. It is defined as the multiplier of the solid angle $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ from (1.74):

$$d_m = S(\chi) \quad (1.75)$$

In case of a flat universe ($k = 0$), the metric distance becomes equal to:

$$d_m = \chi \quad (1.76)$$

The second one is the *comoving distance*, which is the distance between two objects remaining constant with epoch, when they move solely due to the expansion of the universe. It can be written as:

$$d_{co} = \int_{t_e}^{t_0} \frac{dt'}{a(t')} = \int_{a_e}^{a_0} \frac{da}{a\dot{a}} = \int_0^z \frac{dz'}{a_0 H(z')} = \frac{1}{a_0 H_0} \int_0^z \frac{dz'}{E(z')} \quad (1.77)$$

with:

$$E(z) \equiv \frac{H(z)}{H_0} \quad (1.78)$$

being the dimensionless Hubble parameter.

1.7.2 Observable distances

The most common observable distance is the *physical distance* r_p . It is the actual proper distance between two objects that can be measured with a ruler. It is related to the comoving distance d_{co} through:

$$r_p = a(t)d_{co} \quad (1.79)$$

Another significant observable distance is the *diameter angular distance*, which is defined as follows:

Suppose there is an object at a comoving distance χ that emitted photons at t_1 . If the object has a known physical size D and someone on Earth measures its angular size to be $\delta\theta$, then the diameter angular distance is defined as:

$$d_A = \frac{D}{\delta\theta} \quad (1.80)$$

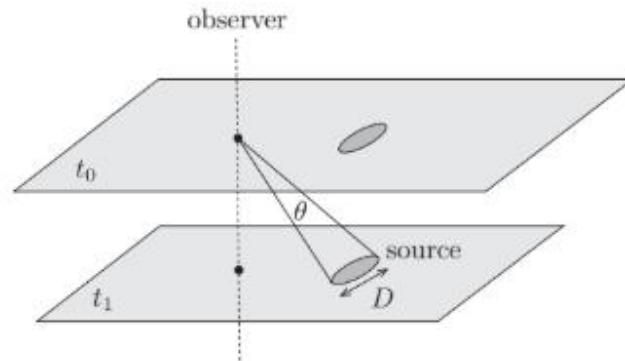


Figure 1.4: Figure about the diameter angular distance (Baumann, D)

which for $\delta\theta \ll 1$ gives the Euclidean formula of its distance.

Through the FRW metric (1.3), the physical transverse size and its angular size on the sky are connected through the relationship:

$$D = a(t_1)S(\chi)\delta\theta \quad (1.81)$$

Combining (1.81) with (1.52), (1.75) we get:

$$d_A = \frac{d_m}{1+z} \quad (1.82)$$

This type of distance is very important because measuring the redshift of objects of known physical size D , which are called “standard rulers”, we can calculate not only the diameter angular distance, but also all the cosmological parameters.

Luminosity distance plays also a vital role. If we assume a source at a fixed comoving distance χ with a known absolute luminosity L (energy emitted per second) then the relation between L and its observed flux (energy per second per area) would be:

$$F = \frac{L}{4\pi\chi^2} \quad (1.83)$$

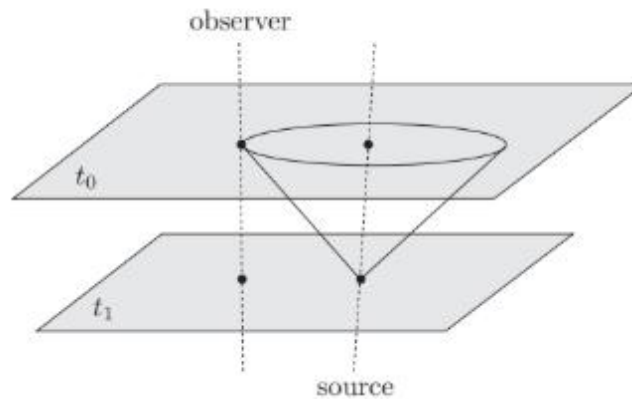


Figure 1.5: Figure about the luminosity distance(Baumann, D)

In other words, the power is distributed in the spherical surface with radius χ . However, the result in FRW metric is modified for three reasons:

- i. The observation happens today at t_0 . As a result, the radius of the sphere drawn around a supernova and reaching the Earth is not χ , is d_m , and its proper area is $4\pi d_m^2$.

- ii. Because of the expansion of the universe and the redshift, the rate of the arriving photons is lower than their emission rate by the factor $1/(1+z)$.
- iii. For the same reason in (ii) the energy rate is also lowered by the factor $1/(1+z)$.

As a result, the relationship between flux and luminosity is:

$$F = \frac{L}{4\pi d_m^2} \frac{1}{(1+z)^2} \quad (1.84)$$

To have a same relationship like in (1.83), the luminosity distance d_L is defined as:

$$d_L = d_m(1+z) \quad (1.85)$$

Luminosity distance is important because of the Standard Candles, i.e objects that have a known absolute luminosity L . Some examples are Cepheids and Type Ia Supernovae.

Astronomers also use the "*magnitude*" system to calculate fluxes and luminosities. (Ryden B., 2017). In ancient Greece, Hipparchus was the first one that divided stars into six classes, according to their apparent brightness. The "first magnitude" consisted of the brightest stars, while the "sixth magnitude" of the faintest ones. Later, it was realized that this categorization was roughly logarithmic, which helped the establishment of the magnitude system on a strict mathematical basis.

The apparent magnitude m of an astronomical object is defined as:

$$m = -2.5 \log_{10} \left(\frac{F}{F_{ref}} \right) \quad (1.86)$$

with F being the apparent flux of the object and F_{ref} a reference flux.

The absolute magnitude M is the apparent flux the object would have if it was 10 pc away from the observer:

$$M = -2.5 \log_{10} \left(\frac{F_{10pc}}{F_{ref}} \right) \quad (1.87)$$

The distance modulus μ is the difference between them and is defined as:

$$\mu \equiv m - M = 5 \log_{10} \left(\frac{d_L}{10 \text{ pc}} \right) \quad (1.88)$$

with d_L being the luminosity distance.

In case of the luminosity distance being expressed in Mpc, then (1.88) is written as:

$$\mu \equiv m - M = 5 \log_{10} d_L + 25 \quad (1.88)$$

CHAPTER 2

Theoretical Background

2.1. Hubble's law

According to contemporary cosmological models the universe is isotropic and homogeneous. This means that the space looks the same regardless the direction and that on a large scale it has a uniform composition and structure everywhere (Carroll M., S., 2019). On a large scale means that these properties apply on scale of roughly 100 Mpc or more (Ryden B., 2017).

One of the most important observational discoveries in cosmology is that nearly all galaxies seem to be moving away from us and the farther a galaxy is, the faster it appears to recede (Figure 2.1). This relationship is described by Hubble's law. Astronomers determine these velocities using redshift measurements, which are based on the Doppler effect for light. Each galaxy's spectrum contains characteristic absorption and emission lines with known frequencies. When a galaxy moves toward us, these light waves are compressed, shifting the spectrum toward higher frequencies, known as a blueshift. When a galaxy moves away, the wavelengths are stretched, shifting the spectrum toward lower frequencies, or a redshift. The Astronomer Edwin Hubble tried to measure the distance to galaxies and his discovery was that recession velocity is proportional to the distance (Liddle A., 2015). This discovery was expressed by Hubble's law (2.1):

$$v = Hr(t) \quad (2.1)$$

where r is the physical distance and H is the Hubble constant.

The physical distance is derived from:

$$r(t) = a(t)x \quad (2.2)$$

where x is the comoving distance and $a(t)$ is the scale factor, which in fact describes how the size of the universe changes with time and at present time is equal to 1.

In (2.1) H is the Hubble parameter and derives from:

$$H(t) = \frac{\dot{a}}{a} \quad (2.3)$$

The present day value of the Hubble is:

$$H_0 = 100 h \text{ kms}^{-1} \text{ Mpc}^{-1}$$

Which means that a galaxy 1 Mpc away retreats at a speed of 70 km/s (Tong D., 2019).

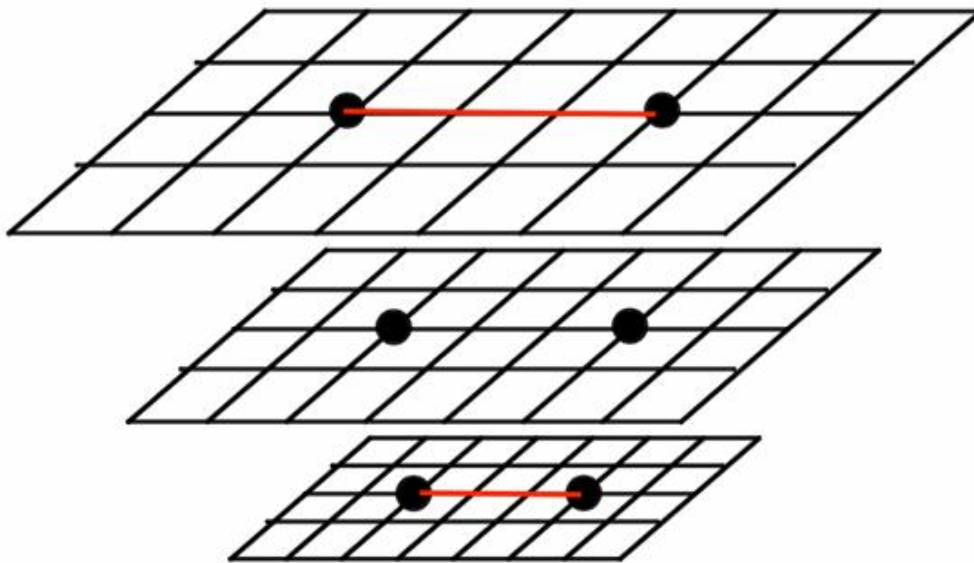


Figure 2.1: The expansion of the universe. The physical distance between fixed co-moving coordinates increases with time (<https://www.sr.bham.ac.uk/~smcgee/ObsCosmo/lectures/ObsCosmo2.pdf>).

2.2. Dark Energy

It is crucial, in order to reconstruct the history of the expansion of the universe $a(t)$, to measure distance by using brightness of an object. For this reason scientists use SN Ia (they will be mentioned later) as standard candles.

From scale factor $a(t)$ Taylor expansion and setting $a_0 = 1$ we have:

$$a(t) = 1 + H_0(t - t_0) - \frac{1}{2}q_0H_0^2(t - t_0)^2 + \dots \quad (2.4)$$

where q_0 is known as the deceleration parameter

$$q_t = -\frac{\ddot{a}}{aH^2} \quad (2.5)$$

which in our universe is negative $q_0 \sim -0.5$.

Considering the universe as a homogeneous and isotropic perfect fluid, it is characterized by energy density $\rho(t)$, pressure $P(t)$ and the relation between them, which is called equation of state:

$$P_m = w_m \rho_m \quad \text{and} \quad P_r = w_r \rho_r \quad \text{with } w_m = 0 \text{ for matter and } w_r = \frac{1}{3} \text{ for radiation.}$$

From Raychaudhuri equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3P) \quad (2.6)$$

we conclude that assuming consisting of matter and radiation, the result would be a decelerating universe since $(\rho + 3P) > 0$. In order to have an accelerating universe we need at least $w < -1/3$. Thus there must be an additional component called "Dark Energy". Eventually from Friedmann Equation for flat universe with only matter and dark energy we get:

$$H^2 = H_0^2 [\Omega_m (1+z)^3 + \Omega_\Lambda] \quad (2.7)$$

where Ω_m and Ω_Λ are the matter density parameter and Dark Energy parameter respectively (Tong D., 2019).

2.3. Supernovae types

Supernovae are stars undergoing a tremendous explosion, which lasts for several months to a few years. Although they are detected once in a few decades, they are observable at very large distances.

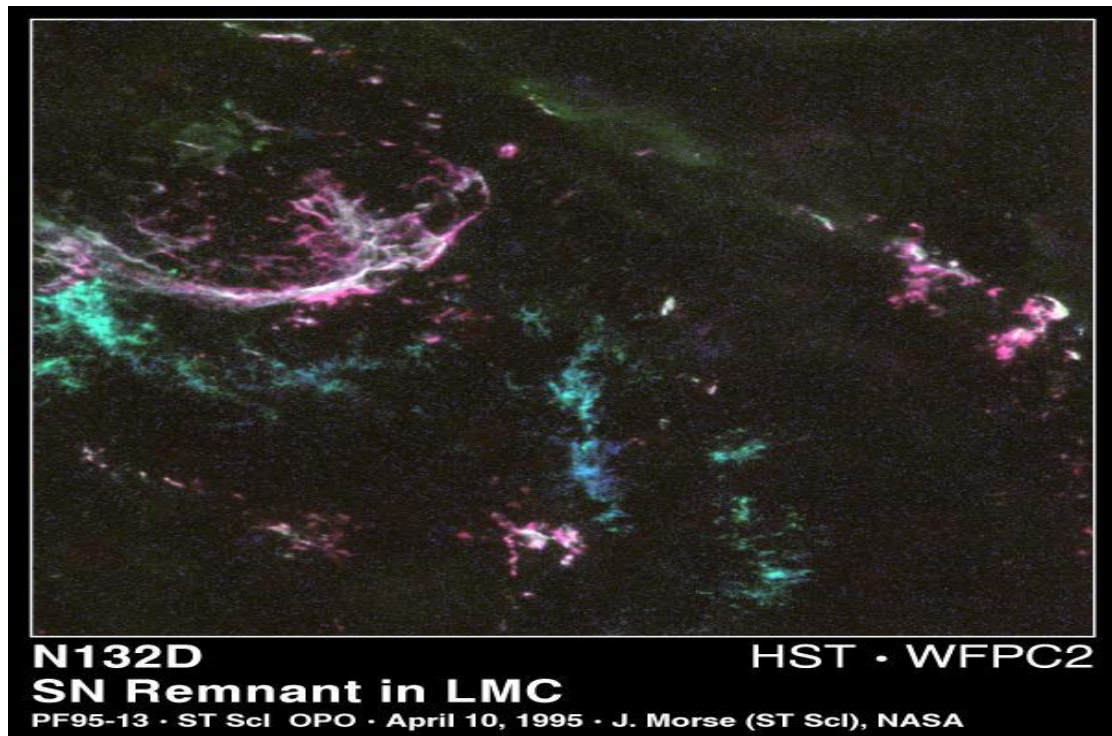


Figure 2.2: Remnants of Supernova N132D, that exploded 3.000 years ago. It is estimated that the progenitor star had a mass of $25 M_\odot$. (<https://www.sr.bham.ac.uk/~smcghee/ObsCosmo/lectures/ObsCosmo2.pdf>).

The nebulae ejected, the so-called remnants, survive for much longer and are some of the most spectacular astronomical objects. Their expansion velocities are very high, up to 10.000 km/s and so they extend rapidly to



remarkable dimensions. Even though they become almost transparent, remain visible for thousands years (Figure 2.2). (<https://www.sr.bham.ac.uk/~smcgee/ObsCosmo/lectures/ObsCosmo2.pdf>).

Supernovae explosions are classified into two types according to their properties:

- Type I- there is absence of hydrogen lines in the spectrum
- Type II- there is presence of hydrogen lines in the spectrum

For Type I the explosion is induced by a thermonuclear runaway in a carbon-oxygen white dwarf, whose mass is $>$ Chandrasekhar mass. Supernovae progenitors can also be binary stars, that consists of two stars revolving around their common center of mass and are born at the same time. Irradiation and tidal distortion have relatively small effect, but the main effect is the transferring material between them, when they come into direct contact.

The problem with the above theory is that during mass transfer, outburst occur and eventually the white Dwarf gradually eroded. Diffusion of elements, turbulent mixing and convection may result in accreted material outburst. Thus it is very hard to reach the Chandrasekhar mass limit, but under special circumstances the white Dwarf will manage to retain part of the accreted mass.

For the single- degenerate model mass transfer can proceed through strong wind from the companion or through Roche- lobe overflow. The companion can be a main sequence star, a sub-giant, a helium star or a red giant.

The Double – Degenerate scenario involves two WDs where the more massive WD accretes the lower massive WD through an accretion disk configuration. According to recent models this procedure, called violent merger, is more complex and may lead to a SN Ia formation even if the total mass of the merger is below M_{ch} . (Figure 2.3).

Right after the explosion there is a shock wave with $u \sim 10^4$ km/s, which reaches the surface of the WD in $t \sim 1$ s. Then there is a gamma- ray flash of energy $E \sim 10^{40}$ erg, called "shock breakout", which lasts for ms. The ejecta then expands and cools adiabatically. After the "shock breakout" the luminosity and temperature decrease, initiating a "dark phase" in the light curve (lasts few days). The radioactive decay of ^{56}Ni in ^{56}Co and then in ^{56}Fe causes the observed rise in the optical light curve, for 18 days, until the peak (Prialnik D., 2010).

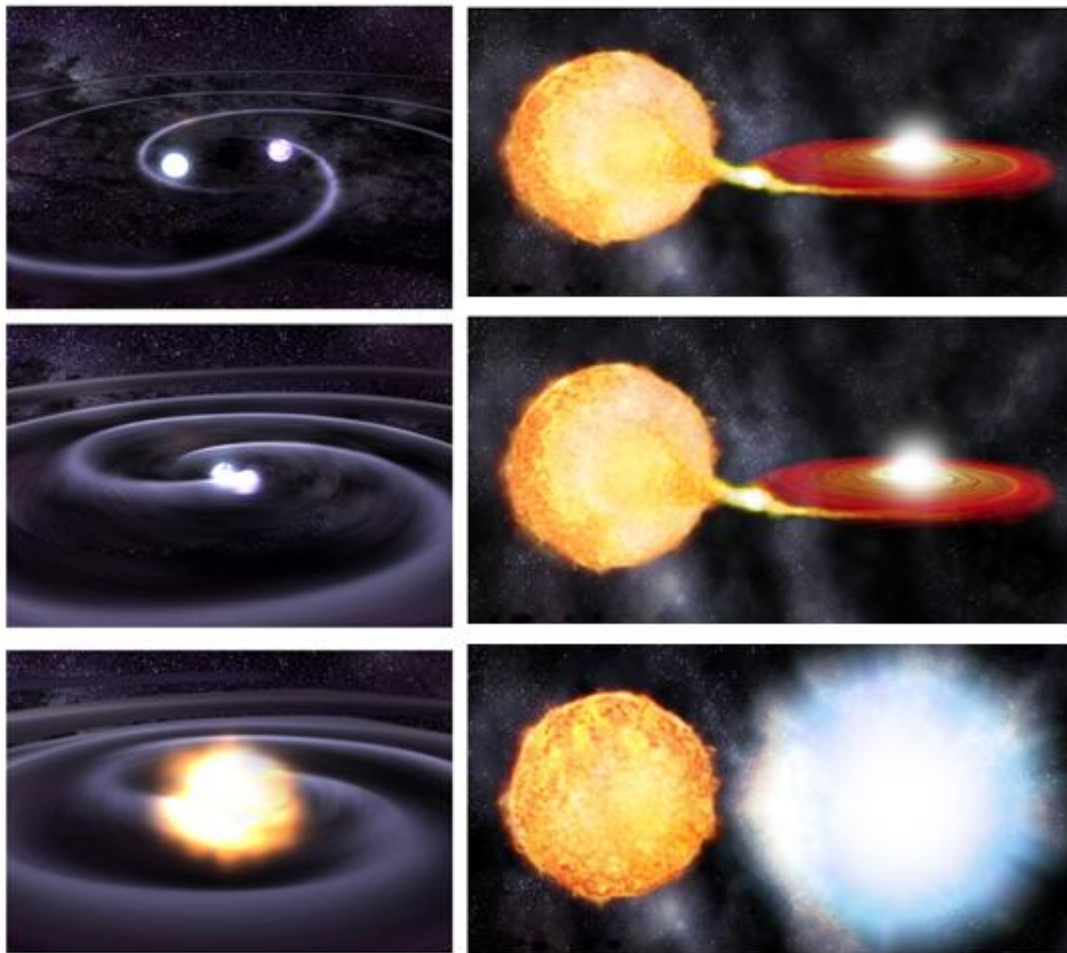


Figure 2.3. Left: the double - degenerate scenario. Right: the single - degenerate mode (Maoz D.,2014)

Light Curve- Observational tool

The luminosity of a SN Ia powered by the decay of the radioactive elements, that are mentioned above and shows a peak, with $0.3-0.8 M_{\odot}$ of ^{56}Fe produced. According to Arnett's rule this peak is proportional to the mass of ^{56}Ni produced in the explosion. The optical light curves (majority of the luminosity emerges at optical wavelengths) are relatively homogeneous with a peak in ~ 20 days and a slow decline before the exponential decay phase after ~ 50 days.

On the other hand the infrared light curves are characterized by a secondary peak 20-30 days after maximum light and the ultraviolet peak comes 15-20 days after explosion, slightly before the optical.

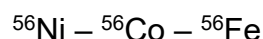
The relationship between luminosity and light curve doesn't show significant dependence on host galaxy properties. It seems that SN Ia in star forming galaxies present more homogeneous population. This is very important because there is no need of using correlations in cosmological analyses, based on host environment. Thus using SN Ia in cosmology is a very good way of measuring dark energy, Hubble constant and universe acceleration.

The spectra reveal the elements produced in the explosion, their location within the SN ejecta and their quantities. At early times the SN Ia spectra are dominated by intermediate-mass elements such as Si, S, Ca, Mg and after maximum, light begins to be dominated by iron-group elements. By ~ 150 days the ejecta expands and becomes optically thin, thus the SN Ia enters the "nebular phase" (Jha S. W., et. al., 2019).

The evolution stages of a SN Ia light curve can be described as follow:

- Shock Propagation and shock Breakout: The thermonuclear explosion begins near the center of the white Dwarf and a shock wave propagates outward at velocities 10^4 km/s. When the shock reaches the surface, a brief gamma-ray flash produced, known as shock breakout. Because this lasts only for milliseconds it is difficult to observe.

- **Shock-Heated Cooling Emission:** The ejecta are extremely hot and expand rapidly and the shock-heated material cools adiabatically. The luminosity decreases, while the emitted radiation shifts from gamma rays and X-rays to ultraviolet and optical wavelengths. This phase lasts minutes to hours after the explosion.
- **Dark Phase:** As the shock-heated emission fades, the luminosity may temporarily fall below detectable levels. The radioactive energy has not yet diffused to the surface and this Dark Phase can last hours to few days, depending on the distribution of radioactive material
- **Radioactive ^{56}Ni -Powered Rise:** Photons produced by the radioactive decay of ^{56}Ni begin to diffuse outward through the ejecta and the optical luminosity rises steadily. In simple models, the luminosity follows $L \propto t^2$, although more realistic calculations predict deviations from this behavior.
- **Peak Luminosity:** The light curve reaches its maximum brightness 18-20 days after the explosion. The radiation from most of the synthesized ^{56}Ni can escape and the peak luminosity is related to the total mass of ^{56}Ni produced during the explosion
- **Post-Peak decline:** The luminosity gradually decreases and this decline is powered by the radioactive decay chain



As the ejecta expand and become more transparent, an increasing fraction of gamma rays escapes without depositing energy.

- **Nebular Phase:** Several months after the explosion the ejecta become optically thin and the spectrum is dominated by emission lines from iron-group elements. The luminosity is mainly powered by the decay of ^{56}Co and by the deposition of positron energy. (Maoz D. et. Al., 2014).

Typical Light Curves of ten well observed SN Ia are shown in Figure 2.4, where the vertical axis represents the distance modulus, i.e. the difference

between the apparent magnitude m and the absolute magnitude M of objects using the relation:

$$\mu = m - M \quad (2.8)$$

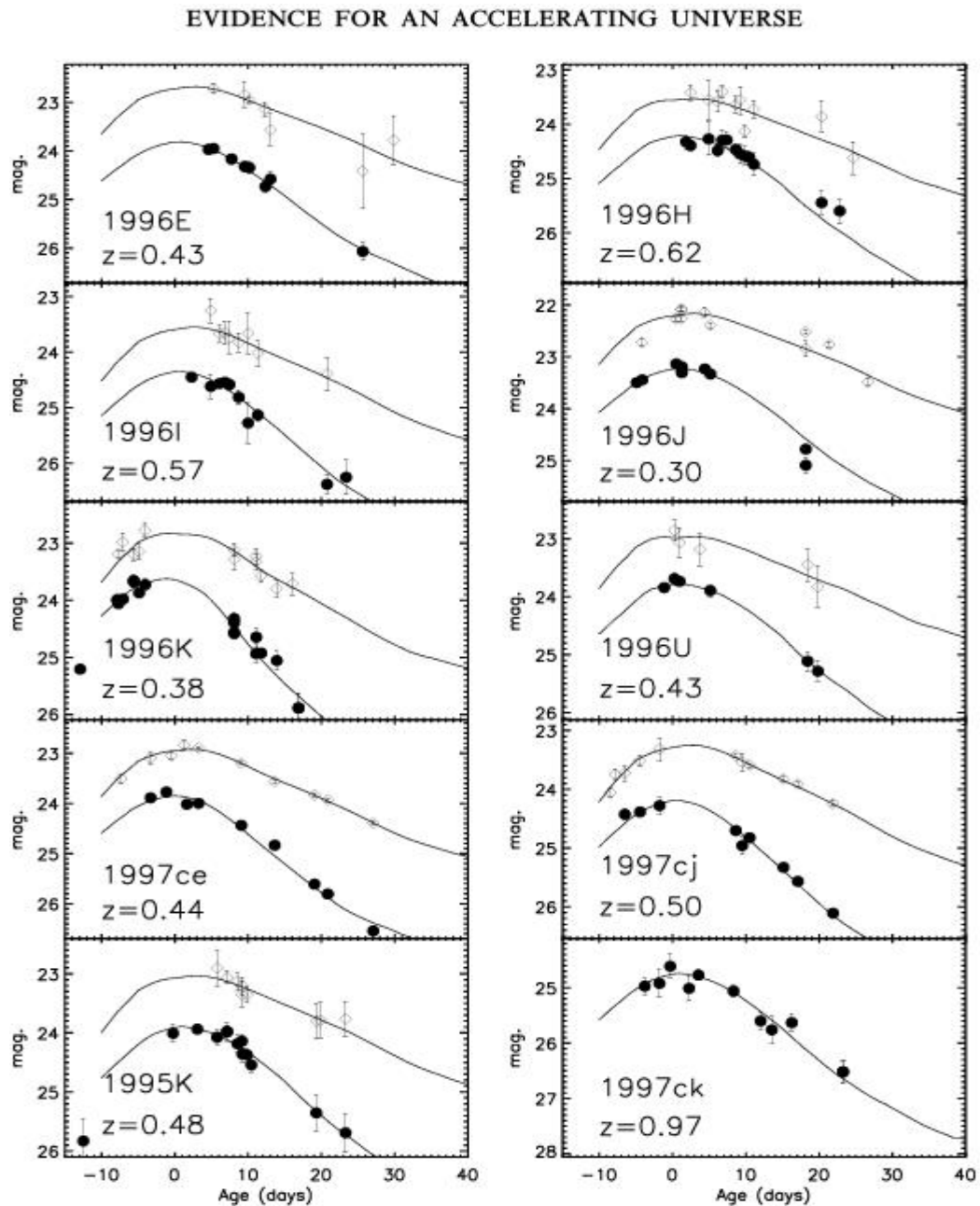


Figure 2.4: Light Curves of 10 well observed SN Ia (Riess A. G. et. al., 2019).



A smaller magnitude corresponds to a higher luminosity, while higher magnitude corresponds to a smaller magnitude. Moreover according to Mark Phillips (1993) brighter Type Ia Supernovae fade more slowly than fainter ones. Measuring Δm_{15} (it measures the decrease in brightness, in magnitudes, 15 days after maximum light) we conclude that a smaller Δm_{15} corresponds to a slower decline in the Light Curve and therefore a higher intrinsic luminosity and vice versa. From (2.8) we calculate the luminosity distance d_L and with the z which derives from SN Ia spectrum we can construct the Hubble diagram. Through this we conclude that as distance increases, Supernovae appear fainter, thereby providing evidence for the accelerating expansion of the universe. (Riess A. G. et. al., 2019).

2.4 Baryonic Acoustic Oscillation (BAO)

Early universe

Early Universe ($1100 \leq z \leq 3400$) consisted of photons, electrons, photons and dark energy in thermal equilibrium and the hydrogen had not yet formed. In the photon- baryon fluid the speed of sound was determined by the photons, so $c \sim \frac{c}{\sqrt{3}}$. The dark matter and the baryonic matter behave differently. This means that the density perturbations in dark matter start to grow, but on the other hand the baryons fall in a region of higher density, while the photons produce a strong radiation pressure which pushes plasma outward. Thus sound waves are created known as Baryonic Acoustic Oscillation (BAO) (Figure 2.5) (Tong D., 2019).

Recombination

When the universe becomes 400.000 years old photons and electrons cooled enough that they could combine to form Hydrogen, which is called recombination.

Photon decoupling

Then photons decoupled from matter which was no longer dragged along by radiation. At the end what was left is a Dark Matter perturbation surrounded by Baryons at the peripheries of the expanding shells.

Galaxies formation

Eventually dark and Baryonic Matter mix, forming galaxies. (Figure 2.5)

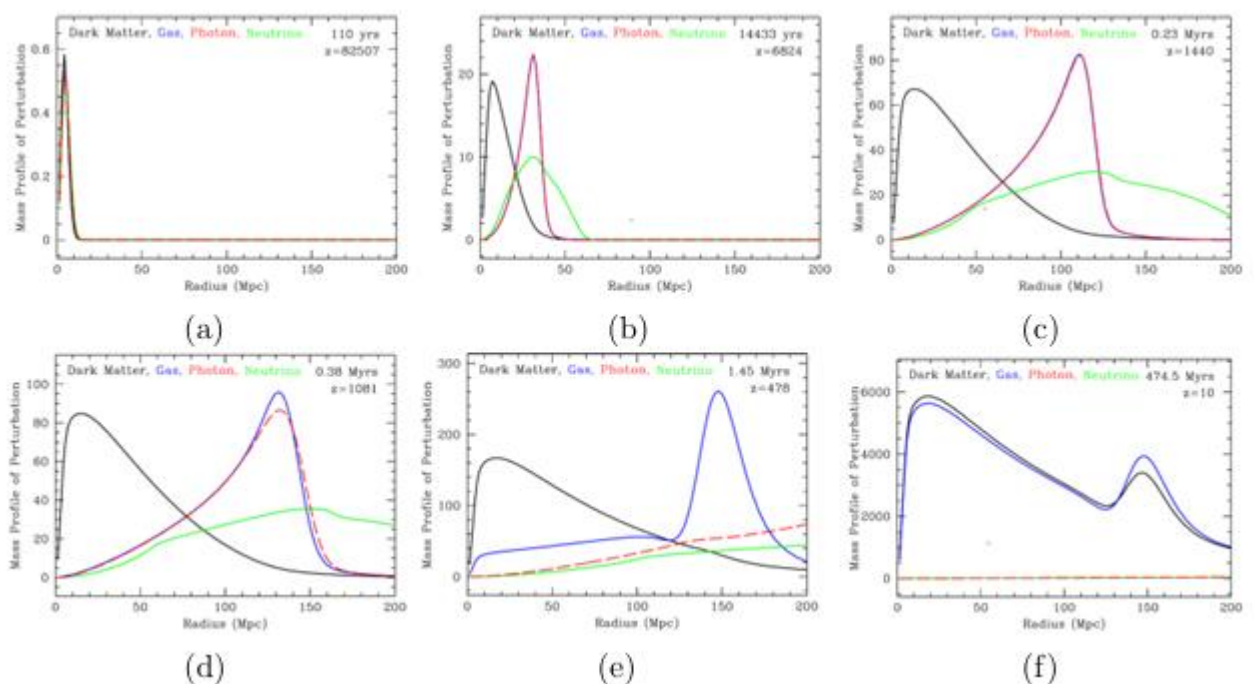


Figure 2.5: The BAO generation description a) early universe b) acoustic waves forming c) Dark Matter perturbation grows d) recombination e) photon decoupling f) Dark Matter and Baryons mix up. (Castander F. J.).

2.5 Drag Epoch and Sound Horizon

After recombination most e have combined with p. As a result, the transfer of momentum from photons to baryons became negligible. However a small fraction of free electrons remained. These electrons continued to couple photons and baryons, allowing a weak transfer of momentum between them. The time that the BAO freeze out is known as Drag Epoch with a redshift:



$$z_d = \frac{1291(\Omega_m h^2)^{0,251}}{1 + 0,659(\Omega_m h^2)^{0,828}} [1 + b_1 (\Omega_b h^2)^{b_2}] \quad (2.9)$$

where:

$$b_1 = 0,313(\Omega_m h^2)^{-0,419} [1 + 0,607(\Omega_m h^2)^{0,674}]$$

$$b_2 = 0,238 (\Omega_m h^2)^{0,223}$$

(Eisenstein J. D and Hu W., 1997).

A characteristic scale that shows the distance the waves have propagated prior to recombination is the Sound Horizon and is given by:

$$r_s = \int_{z_d}^{\infty} \frac{c_s}{H(z)} dz \quad (2.10)$$

where :

$$c_s = \frac{c}{\sqrt{3(1 + R_s)}} \quad (2.11)$$

is the sound speed that Photon-Baryon fluid acoustic waves propagate.

In (2.11):

$$R_s = \frac{3\rho_b}{4\rho_\gamma} = \frac{3\Omega_b}{4\Omega_\gamma} \frac{1}{1+z} \quad (2.12)$$

means that the more the Baryons the less the sound speed is. As we can see from (2.10) the Sound Horizon depends from expansion of the universe, z_d and the Baryon-to-Photon ratio (Addison G. E., 2013).

2.6 BAO measurements

In a spectroscopic galaxy survey, we do not directly measure distances; instead we measure galaxy redshift. The BAO feature provides a characteristic scale r_s , which acts as a known cosmological standard ruler. If two galaxies are separated along the line of sight by the BAO scale, this physical separation corresponds to a measurable redshift Δz . For small intervals, the relation between comoving distance and redshift is:

$$\Delta z = \frac{r_s H(z)}{c} \quad (2.13)$$

Helping us measure the expansion rate $H(z)$ Hubble distance :

$$D_H = \frac{c}{H(z)} \quad (2.14)$$

quantifies the characteristic distance scale associated with the expansion rate of the universe at a given redshift.

The BAO signal can also be observed perpendicular to the line of sight, where it appears as a characteristic angular scale on the sky. The observed angular extent of the BAO feature derives from:

$$\theta_s = \frac{r_s}{(1+z)D_A(z)} \quad (2.15)$$

So by measuring the angular extent of the BAO feature and using the known sound horizon scale r_s we determine $D_A(z)$ and eventually the comoving angular diameter $D_m(z)$ (Alam S., et al.) from:

$$D_m(z) = (1+z)D_A(z) \quad (2.16)$$

2.7 CMB Measurements

CMB is the thermal radiation and the first piece of evidence for the hot Big Bang. Measures the same physical scale r_s , but at the very high redshift of recombination ($z \sim 1090$). The characteristic angle of the location of the peaks is defined from:

$$\theta_A = \frac{r_s(z_*)}{D_m(z_*)} \quad (2.17)$$

that means it depends on the sound horizon and the distance until the last scattering surface.

The multipole l (small l corresponds to large angular-scale structure) corresponding to the angle θ_A is:

$$l_A = \frac{\pi}{\theta_A} = \pi \frac{D_m(z_*)}{r_s} \quad (2.18)$$

as we can see bigger D_M means smaller angle and bigger multipole l .

Parameter R is perhaps the least model-dependent parameter, since it is independent of H_0 and defined as:

$$R = \frac{\sqrt{\Omega_m H_0^2}}{c} D_m(z_*) \quad (2.19)$$

It is very important since it summarizes the geometric information of the CMB.

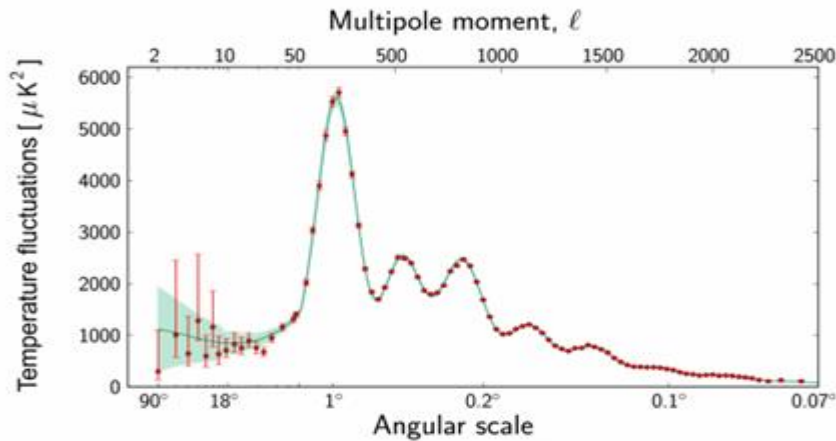


Figure 2.6: CMB power spectrum measured by Planck (Tong D., 2019).

The CMB power spectrum (Figure 2.6) is a characteristic peaks and troughs that originate from oscillations in the photon-baryon plasma before recombination.

The first peak's location depends on the physical scale of the sound horizon and the distance to the surface of last scattering. It is estimated from:

$$l_{peak} = \frac{\pi}{\theta_s} \sim 200$$



The fact that $l \sim 200$ proves that the universe is almost flat $\Omega_k \sim 0$. Measurements estimate that $\Omega_m + \Omega_r < 1$ and there was not any other form of energy the universe could not be flat. So the location of the first peak proves the existence of another energy $\Omega_\Lambda \sim 0,7$ called dark energy.

The second and third peaks contain information about the baryon and dark matter densities, since their amplitudes depend on the baryon-to-photon ratio and the gravitational potentials generated by dark matter. This means that measurements of the CMB make it possible to determine the densities of baryonic matter and dark matter independently (Tong D., 2019).

Chapter 3

Data Analysis

Most cosmological analysis data problems are inverse problems. We have a set of data $\mathbf{x}$ which we want to interpret in a certain way. These problems are classified to *parameter estimation* and *model selection*.

Parameter estimation

3.1 Bayes Theorem

In this category, the data are collected and interpreted in terms of a *model*, a theoretical framework that is believed to be true. The goal of the parameter estimation is the determination of the probability distribution of the parameters θ given the data $\mathbf{x}$. This is called the *posterior probability distribution*. It represents the probability that the parameters take certain values after doing the experiments and is estimated from *Bayes Theorem* which states that (Caroll S. M., 2019):

$$p(\theta|\mathbf{x}) = \frac{p(\mathbf{x}|\theta)p(\theta)}{p(\mathbf{x})} \quad (3.1)$$

where:

- $p(\theta|\mathbf{x})$ is the posterior probability distribution of the parameters θ given the data $\mathbf{x}$
- $p(\mathbf{x}|\theta)$ is the likelihood and has its own symbol $\mathcal{L}(\mathbf{x}; \theta)$. It is the probability distribution of data $\mathbf{x}$ if the parameters θ were right.
- $p(\theta)$ is called the *prior*. It expresses the probability distribution of the parameters θ before the experiments. All we know about the parameters are results from previous experiments or theory. For example, the age of the Universe must be positive. In the absence of any previous information it is common to adopt the principle of indifference and assume that all values of the parameters are equally likely and take $p(\theta) = \text{constant}$. As bounds someone can use finite bounds or infinite bounds with unnormalized prior. This is called *Flat Prior*.

- $p(x)$ is the *evidence* and is used for normalization.

3.2 χ^2 function

3.2.1 Independent measurements

The likelihood is expressed in terms of the χ^2 function, since there are many observations and the errors are under Gaussian distribution because of the Central Limit Theorem.

Assume there are data with M independent measurements $X_{i,obs}$ with known standard deviation σ_i . If we have a theoretical model $X_{th}(p)$ we want to test with the data - with p representing the parameters used in the model - , then the χ^2 function is defined as (Liddle A. R., 2015):

$$\chi^2(p) = \sum_{i=1}^M \frac{[X_{th}(p) - X_{i,obs}]^2}{\sigma_i^2} \quad (3.2)$$

In other words, this function quantifies the discrepancy between the predictions of the theoretical model and the observations at a particular value of the parameters p. A good fit is defined as a fit which gives small value of χ^2 . The values p_0 that minimize the function are called best fit parameters.

3.2.2 Non - independent measurements

If X and Y are two random variables, then their *covariance*, denoted as $cov(X, Y)$ is equal to [4]:

$$cov(X, Y) = \langle (X - \langle X \rangle)(Y - \langle Y \rangle) \rangle \quad (3.3)$$

where $\langle X \rangle$, $\langle Y \rangle$ are the mean values of X and Y respectively.

For discrete random variables, the covariance is:

$$cov(X, Y) = \sigma_{ij} = \sum_{i,j} (x_i - \langle x_i \rangle)(y_j - \langle y_j \rangle) h(x_i, y_j) \quad (3.4)$$

with $h(x_i, y_j)$ being the joint distribution of X, Y.

For continuous random variables:

$$cov(X, Y) = \int dx \int dy (x - \langle X \rangle)(y - \langle Y \rangle) f(x, y) \quad (3.5)$$

with $f(x, y)$ being the joint distribution of X, Y .

Thus,

$$cov(X, X) = \langle (X - \langle X \rangle)(X - \langle X \rangle) \rangle = \langle (X - \langle X \rangle)^2 \rangle = Var(X) = \sigma_X^2 \quad (3.6)$$

The *correlation* of the two variables is defined as:

$$\rho(X, Y) = \frac{cov(X, Y)}{\sigma_X \sigma_Y} \quad (3.7)$$

In the case of discrete variables the correlation takes the following form:

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} \quad (3.8)$$

For the measurement vector X_i we construct the *covariance matrix*:

$$V_{ij} = cov(X_i, X_j) = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1N}\sigma_1\sigma_N \\ \rho_{21}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \rho_{2N}\sigma_2\sigma_N \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N1}\sigma_N\sigma_1 & \rho_{N2}\sigma_N\sigma_2 & \cdots & \sigma_N^2 \end{bmatrix} \quad (3.9)$$

Thus, the covariance matrix is always square and symmetric.

Since the variables are non - independent, the χ^2 function takes the form of (Liddle A. R., 2015):

$$\chi^2(p) = [X_{th}(p) - X_{i,obs}]^T V^{-1} [X_{th}(p) - X_{i,obs}] \quad (3.10)$$

3.2.3 “Chi - by - eye”

The “chi - by - eye” rule states that in a good fit the minimum χ^2 is roughly equal to the number of data minus the number of parameters. That is true for a large number of data. In other words, if $m \gg n$ (m the number of data and n the number of parameters, then the reduced chi-square is (McGee S.) :

$$\chi_v^2 = \frac{\chi_{min}^2}{m} \approx 1 \quad (3.11)$$

3.3 Likelihood function

The likelihood function is defined as (Liddle A. R., 2015):

$$\mathcal{L}(p) = \exp\left[-\frac{1}{2}\chi^2(p)\right] \quad (3.12)$$

The likelihood function maximizes at the set of parameters p_0 that χ^2 minimizes. Values of the parameters that result in higher value of the likelihood function are more likely to be the values of the true parameters. If there are two parameters, the likelihood function can be visualized as a surface sitting above the parameter space.

3.3.1 Fisher Matrix

It is important to estimate the errors of the likelihood function. At first, we work with a flat prior so that the posterior is identified through the likelihood. Then, the likelihood is expanded as Taylor series around its maximum (McGee S.):

$$\begin{aligned} \ln \mathcal{L}(p) = \ln \mathcal{L}(p_0) &+ \sum_{j=1}^N \left. \frac{\partial \ln \mathcal{L}(p)}{\partial p_j} \right|_{p_0} (p_j - p_{j,0}) \\ &+ \frac{1}{2!} \sum_{i=1}^N \sum_{j=1}^N \left. \frac{\partial^2 \ln \mathcal{L}(p)}{\partial p_i \partial p_j} \right|_{p_0} (p_i - p_{i,0})(p_j - p_{j,0}) + \dots \quad (3.13) \end{aligned}$$

The derivative of the log - likelihood is equal to zero for every p_0 since p_0 is the maximum:

$$\left. \frac{\partial \ln \mathcal{L}(p)}{\partial p_j} \right|_{p_0} = 0 \quad (3.14)$$

The expansion stops to the quadratic term, so the likelihood surface is a multivariate Gaussian. As a result:

$$\ln \mathcal{L}(p) \approx \ln \mathcal{L}(p_0) + \frac{1}{2!} \sum_{i=1}^N \sum_{j=1}^N \left. \frac{\partial^2 \ln \mathcal{L}(p)}{\partial p_i \partial p_j} \right|_{p_0} (p_i - p_{i,0})(p_j - p_{j,0}) \quad (3.15)$$

Hessian matrix

The Hessian matrix is defined as (Liddle A. R., 2015):

$$\mathcal{H}_{ij} = - \frac{\partial^2 \ln \mathcal{L}(p)}{\partial p_i \partial p_j} \quad (3.16)$$

The Hessian matrix encloses information about the parameters errors and their covariance. A non - diagonal matrix implies that the parameters are correlated. In this case, they have a same effect on the data and it's difficult to discern them. The parameters, though, may not be physically correlated.

Conditional errors

Assume that all parameters are fixed, except one, which is defined as p_i . Then, its error is equal to (Caroll S. M., 2019):

$$\sigma_{p_i} = \frac{1}{\sqrt{\mathcal{H}_{ii}}} \quad (3.17)$$

This is called *conditional error* and is the minimum error bar attainable on p_i when the other parameters are fixed, but it is uninteresting and rarely used.

Fisher matrix

In 1935 Fisher answered the question of how accurately can someone measure model parameters from a given data set. The Fisher matrix is defined as (Prialnik D., 2010):

$$F_{ij} = \left\langle \frac{\partial^2 L(p)}{\partial p_i \partial p_j} \right\rangle \quad (3.18)$$

where $L = -\ln \mathcal{L}(p)$, or as:

$$F_{ij} = \langle \mathcal{H} \rangle \quad (3.19)$$

We choose the fiducial model and compute the Fisher matrix at the fiducial model. Around p_0 we have:

$$\Delta \ln \mathcal{L} = \ln \mathcal{L}(p) - \ln \mathcal{L}(p_0) \quad (3.20)$$

Combining (3.15) and (3.20) and assuming one parameter case we get:

$$\Delta \ln \mathcal{L} = \frac{1}{2} F_{ii} (p_i - p_{i,0})^2 \quad (3.21)$$

Since $\mathcal{L} = e^{-\frac{\chi^2}{2}}$, then we get that $2\Delta \ln \mathcal{L}$ is equal to $\Delta \chi^2$, we see that the displacement 1σ for the parameter p_i when $2\Delta \ln \mathcal{L}=1$ is:

$$\sigma_{p_i} = \frac{1}{\sqrt{F_{ii}}} \quad (3.22)$$

(3.22) is analogous to the conditional error. In general:

$$\sigma_{ij}^2 \geq (F^{-1})_{ij} \quad (3.23) \quad \sigma_{p_i} \geq (F^{-1})_{ii}^{1/2} \quad (3.24)$$

In the last two equations, F^{-1} is the inverse of the Fisher matrix, while σ_{p_i} is the marginalized error, which will be explained later.

The Fisher matrix gives inequalities that take the form of Cramér - Rao Inequality. They become equal only when the likelihood is Gaussian. (Prialnik D., 2010).

3.4 Marginalization

In most of the cases, some parameters of the model are totally uninteresting. We may also have nuisance parameters, which are parameters whose values are known with limited accuracy. An example of a nuisance parameter is the Hubble constant H_0 in dark energy models.

In other cases, we are interested on calculating the constraints on one parameter at that time rather than calculating the joint constraints of two or more parameters simultaneously. For these reasons, we use the technique of marginalization.

Assume we have parameter p and nuisance parameter v with some estimated prior distribution $\pi(v)$. A reasonable choice is the Gaussian probability density function, where v_0 and σ_v are the mean value and variance, respectively. (Prialnik D., 2010).

After that, the posterior likelihood function that depends only on parameter p is:

$$\mathcal{L}(p) = \int \mathcal{L}(p, v) \pi(v) dv \quad (3.25)$$

Considering $\pi(v)$ to be Gaussian, we get:

$$\mathcal{L}(p) = \frac{1}{\sqrt{2\pi\sigma_v^2}} \int \mathcal{L}(p, v) \exp\left[-\frac{(v - v_0)^2}{2\sigma_v^2}\right] dv \quad (3.26)$$

Again the best fit parameters are those who give maximum likelihood.

3.5 Confidence limits

Since there is too much information about the probability distribution of the uncertainties in parameter estimation, we choose to summarize them in the form of *confidence limits*. The full probability distribution is defined in the M -dimensional space of the parameters p , with M being the number of parameters.

A *confidence region* is defined as the region of this space that contains a certain percentage of the total probability distribution. The aim is the confidence region to be as small in the space as possible and to contain as large percentage of the distribution as possible.

During the analysis, we are free to pick the confidence level and the confidence region. The only requirement is that the region does include the certain percentage of the probability. However, in scientific usage the certain percentages are 68.3%, 95.4%, 99.73%, which correspond to standard deviations of 1σ , 2σ , 3σ from the most likely value, respectively. The shape of the region is compact and centered around the measured value. For 1 dimension it is a line segment centered around the measured value, while for 2 or more dimensions ellipses or ellipsoids are chosen.

The meaning of the confidence level is to inspire confidence in that measured value. For example, a region with confidence level 99% means that there is 99% chance that the true parameter falls within this region around the measured value (Maoz D. et al., 2014).

3.5.1 Constant chi-square boundaries as confidence limits

We use the value $p = p_0$ for the parameters in order to minimize the value of χ^2 for the observed data set. Its value increases when the vector of parameters' value p is perturbed away from p_0 . The region within which χ^2 increases by no more than a set amount $\Delta\chi^2$ defines some M - confidence region around p_0 . The Confidence region is large if the $\Delta\chi^2$, otherwise, for a small value, it is small.

We can use any value for $\Delta\chi^2$, but there are values that correspond to characteristic Confidence Limits, like 68.3%, 95.4%, 99.73%. The values depend not only on the Confidence Limits, but also on the number M of parameters (Maoz D. et al., 2014).

3.5.2 Errors

When we use a model it is important to know not only the favoured values according to the given data set, but also their errors. We use a simple model, the linear regression:

$$y = a + bx \quad (3.27)$$

with errors in both axes $\sigma_{x_i}, \sigma_{y_i}$.

To calculate χ^2 , we need at first to calculate the weighted sum of the variances. In this case, the variance is that of the linear combination $y_i - a - bx_i$ of the random variables x_i, y_i .

$$Var(y_i - a - bx_i) = Var(y_i) + b^2 Var(x_i) = \sigma_{y_i}^2 + b^2 \sigma_{x_i}^2 \equiv \frac{1}{w_i} \quad (3.28)$$

As a result:

$$\chi^2(a, b) = \sum_{i=0}^{N-1} w_i (y_i - a - bx_i)^2 = \sum_{i=0}^{N-1} \frac{(y_i - a - bx_i)^2}{\sigma_{y_i}^2 + b^2 \sigma_{x_i}^2} \quad (3.29)$$

This function will give maximum likelihood parameter estimations for a, b if the measurement errors are normally distributed. To find the best fit parameters for a, b we need to solve:

$$\frac{\partial \chi^2}{\partial a} = 0 \quad (3.30) \quad \frac{\partial \chi^2}{\partial b} = 0 \quad (3.31)$$

A 1σ region is defined when: $\chi^2 = \chi_{min}^2 + 4\chi^2$, with χ_{min}^2 being the value of the best - fit parameter and $\Delta\chi^2 = 1$. The standard errors of a and b are calculated as the projections onto the a - and b -axes of the "confidence region boundary".

The projections are calculated through the Taylor expansion of χ^2 :

$$\begin{aligned}\chi^2(a + \Delta a, b + \Delta b) &= \chi^2(a, b) + \frac{\partial \chi^2}{\partial a}(a + \Delta a - a) + \frac{\partial \chi^2}{\partial b}(b + \Delta b - b) \\ &+ \frac{1}{2!} \frac{\partial^2 \chi^2}{\partial a^2}(a + \Delta a - a)^2 + \frac{1}{2!} \frac{\partial^2 \chi^2}{\partial b^2}(b + \Delta b - b)^2 \\ &+ \frac{\partial^2 \chi^2}{\partial a \partial b}(a + \Delta a - a)(b + \Delta b - b) + \dots\end{aligned}\quad (3.32)$$

Using (3.30) and (3.31), we get:

$$\Delta \chi^2 \approx \frac{1}{2} \left[\frac{\partial^2 \chi^2}{\partial a^2} (\Delta a)^2 + \frac{\partial^2 \chi^2}{\partial b^2} (\Delta b)^2 \right] + \frac{\partial^2 \chi^2}{\partial a \partial b} (\Delta a)(\Delta b) \quad (3.33)$$

Since it's difficult to use analytic formulas for second derivatives and the low - order term gives poor approximation of $\Delta \chi^2$, we solve numerically the equation $\Delta \chi^2 = 1$. Thus, we calculate the components from the Fisher matrix and, consequently, the marginal errors of the parameters.

3.6 Monte Carlo

3.6.1 Monte Carlo Markov Chain (MCMC) method

The aim of this procedure is to produce a set of points in the parameter space whose distribution function is the same as the target density, which in this case is the likelihood, and, more generally, the posterior. In this method, we make random drawings in the parameter space, where the next sample depends only on the present one. The resulting Markov chain of points samples the posterior, such that the density of points is proportional to the target density. Thus, we can estimate all the usual quantities, like the mean and the variance. The number of required points is set to scale linearly with the number of parameters. In cosmology, MCMC is an effective tool, since we usually have to deal with 10 - 20 parameters.

The target density is approximated by a set of delta functions (Caroll S. M., 2019):

$$p(\theta) \simeq \frac{1}{N} \sum_{i=1}^N \delta(\theta - \theta_i) \quad (3.34)$$

From this equation we can estimate any integrals:

$$\langle f(\theta) \rangle = \frac{1}{N} \sum_{i=1}^N f(\theta_i) \quad (3.35)$$

The concept of the MCMC is to go from a point θ in the parameter space to a new point θ^* and accept it in the chain with a probability density depending on the ratio of the old and new target densities.

The distribution of steps is called proposal distribution. The most common algorithm is the Metropolis - Hastings algorithm, according to which the probability of accepting the new point as a new point in the chain is defined as:

$$p(\text{acceptance}) = \min \left[1, \frac{p(\theta^*)q(\theta^*|\theta)}{p(\theta)q(\theta|\theta^*)} \right] \quad (3.36)$$

where $q(\theta^*|\theta)$ is the proposal distribution function to move from θ^* to θ .

In case of a symmetric distribution function, (3.36) is simplified to:

$$p(\text{acceptance}) = \min \left[1, \frac{p(\theta^*)}{p(\theta)} \right] \quad (3.37)$$

Thus, the procedure is choosing a random initial point, computing its target density, then moving to the next point with a step generated from a proposal distribution. After we compute the target density of the new point, we accept it using the Metropolis - Hastings parameter. If it's accepted, it becomes the new point in the chain. If it's not, we go back to the previous point.

3.6.2 Hamiltonian Monte Carlo

In the MCMC method we have to select the proper proposal distribution [5]. If it is small, then it will take much time to explore the target distribution, thus the method is inefficient. On the other hand, if it is too large, then although the parameter space is explored, the trial points are far from the peak. The target density is low at those points, so almost all of them are rejected after the Metropolis - Hastings algorithm.

Hamiltonian Monte Carlo tries to deal with these problems. In other words, it takes big steps, so that the chain is well - mixed and simultaneously, the target density of the chosen points is high.

We work as follows:

If the target density is $p(\theta)$, we define the potential:

$$U(\theta) \equiv -p(\theta) \quad (3.38)$$

For each coordinate θ_a we generate a momentum u_a , usually from a distribution with zero mean and unit variance, so the M - dimensional momentum distribution is a simple multivariate Gaussian $N(u)$. The kinetic energy is defined as:

$$K(u) \equiv \frac{1}{2} u^T u \quad (3.39)$$

and the Hamiltonian as:

$$H(\theta, u) \equiv U(\theta) + K(u) \quad (3.40)$$

The extended target density is:

$$p(\theta, u) = \exp[-H(\theta, u)] \quad (3.41)$$

Since it's separable

$$p(\theta, u) = \exp[-U(\theta)] \exp[-K(u)] \propto p(\theta) N(u) \quad (3.42)$$

if we marginalize over u coordinates, the resulting marginal distribution is the target distribution $p(\theta)$. If we exactly solve the Hamiltonian equations:

$$\dot{\theta}_a = u_a \quad (3.43)$$

$$\dot{u}_a = -\frac{\partial H}{\partial \theta_a} \quad (3.44)$$

then H is invariant, the target density is always the same and the acceptance is unity. However, there are two approximations, one which is about the potential calculation and the other about calculating the equations with a

simple integrator to generate a new sample. Thus, H is not invariant, so we use the Metropolis algorithm to accept the new point (θ^*, u^*) :

$$p(\text{acceptance}) = \min\{1, \exp[-H(\theta^*, u^*) + H(\theta, u)]\} \quad (3.45)$$

Model selection

Model selection is a higher - level question than parameter estimation. In parameter estimation there is a theoretical model within which one interprets the data. On the other hand, in model selection one wants to know which theoretical framework is preferred, given the data. (Carroll S. M., 2019).

A model with more parameters gives smaller value of χ^2 , but the model becomes more complex. The goal of model selection is to balance the quality of fit to observational data against the complexity, or predictiveness, of the model achieving that fit. (Martel L. M. V., 2000):

3.7 AIC/BAIC

The most known tools of model selection are the *Akaike Information Criterion (AIC)* and the *Bayesian Information Criterion (BIC)*. The AIC was named after Akaike in 1974. It measures the difference between true data distribution and model distribution and is defined as (Martel L. M. V., 2000):

$$AIC \equiv -2\ln\mathcal{L}_{\max} + 2k \quad (3.46)$$

where $\mathcal{L}_{\max}$ is the maximum likelihood achieved by the model and k the number of parameters of this model. The best model is the one which minimizes the AIC, while the models don't have to be nested.

The BIC was introduced by Schwartz in 1978. It comes from the evidence ratios of models, known as Bayes factor. It is defined as:

$$BIC \equiv -2\ln\mathcal{L}_{\max} + k\ln N \quad (3.47)$$

with N being the number of data-points used in the fit. The BIC assumes that the data-points are independent and identically distributed, which may be or may not be valid, something that depends on the dataset under consideration.

3.8 χ^2 for CMB data

The data that we will use are from Planck 2018 (Villar V., 2015) and are the central values of R , l_A and $\omega_b = \Omega_b h^2$. The data are expressed in terms of a vector and a covariance matrix. The vector is:

$$v = \begin{pmatrix} 1.74963 \\ 301.80845 \\ 0.0237 \end{pmatrix} \quad (3.48)$$

and the covariance matrix is:

$$C_v = 10^8 \times \begin{pmatrix} 1598.9554 & 17112.007 & -36.311179 \\ 17112.007 & 811208.45 & -494.79813 \\ -36.311179 & -494.79813 & 2.1242182 \end{pmatrix} \quad (3.49)$$

We construct the vector:

$$vec = v_{th} - v_{obs} = \begin{pmatrix} R - 1.74963 \\ l_A - 301.80845 \\ \Omega_b h^2 - 0.0237 \end{pmatrix} \quad (3.50)$$

Thus, the χ^2 function is defined as:

$$\chi^2_{CMB} = vec^T \cdot C^{-1}_{CMB} \cdot vec \quad (3.51)$$

3.9 χ^2 for BAO data

The data that we will use are from 6dFGS and WiggleZ (Jha S. W. et. al., 2019) from SDSS (Castander F. J., 2008), (Eisenstein D. J. & Hu W., 1998) and Ly- α measurements (Addison G. E., et. al., 2013).

3.9.1 6dFGS and WiggleZ

These two surveys give measurements for the ratio of:

$$d_z \equiv \frac{r_s(z_d)}{D_V(z)} \quad (3.52)$$

From 6dFGS we get: $d_z(z = 0.106) = 0.336 \pm 0.015$, while WiggleZ gives the three correlated measurements:

$$d_z(z = 0.44) = 0.073, \quad d_z(z = 0.6) = 0.0726, \quad d_z(z = 0.73) = 0.0592$$

The inverse of the covariance matrix is given as:

$$C^{-1}_{WiggleZ} = \begin{pmatrix} 1040.3 & -807.5 & 336.8 \\ -807.5 & 3720.3 & -1551.9 \\ 336.8 & -1551.9 & 2914.9 \end{pmatrix} \quad (3.53)$$

From the four measurements we construct the vector:

$$v = \begin{pmatrix} z \\ d_z \\ \sigma_{d_z} \end{pmatrix} = \begin{pmatrix} 0.106 \\ 0.336 \\ 0.015 \end{pmatrix}, \begin{pmatrix} 0.44 \\ 0.073 \\ 0.031 \end{pmatrix}, \begin{pmatrix} 0.6 \\ 0.0726 \\ 0.0164 \end{pmatrix}, \begin{pmatrix} 0.73 \\ 0.0592 \\ 0.0185 \end{pmatrix} \quad (3.54)$$

Thus, the total inverse covariance matrix is:

$$C^{-1} = \begin{pmatrix} (\frac{1}{0.015})^2 & 0 & 0 & 0 \\ 0 & 1040.3 & -807.5 & 336.8 \\ 0 & -807.5 & 3720.3 & -1551.9 \\ 0 & 336.8 & -1551.9 & 2914.9 \end{pmatrix} \quad (3.55)$$

The vector used to define χ^2 is defined as:

$$vec = d_z^{th} - d_z(z) \quad (3.56)$$

Then the χ^2 function is defined as:

$$\chi^2_{6dFGS,WiggleZ} = \text{vec}^T \cdot C^{-1} \cdot \text{vec} \quad (3.57)$$

3.9.2 SDSS

This survey gives measurements of:

$$\frac{D_V}{r_d} \times r_d^{fid} \quad (3.58)$$

where:

$$r_d = r_s(z_d) \quad (3.59)$$

and its value in the context of fiducial cosmology is:

$$r_d^{fid} = 149.28 \text{ Mpc} \quad (3.60)$$

The data vectors are:

$$v = \begin{pmatrix} z \\ \frac{D_V}{r_d} \\ \sigma \end{pmatrix} = \begin{pmatrix} 0.15 \\ 4.465666824 \\ 0.1681350461 \end{pmatrix}, \begin{pmatrix} 0.32 \\ 8.4673 \\ 0.167 \end{pmatrix}, \begin{pmatrix} 0.57 \\ 13.7728 \\ 0.134 \end{pmatrix} \quad (3.61)$$

The $\frac{D_V}{r_d}$ is the ratio $\frac{1}{d_z}$. Thus, the χ^2 function is:

$$\chi^2_{SDSS} = \sum_{i=1}^3 \frac{\left[\frac{1}{d_z(z_i)} - \left(\frac{D_V}{r_d} \right)_i \right]^2}{\sigma_i^2} \quad (3.62)$$

3.9.3 Ly- α

From this survey the data vectors are:

$$v_1 = \begin{pmatrix} z \\ \frac{D_M}{r_d} \\ \sigma \end{pmatrix} = \begin{pmatrix} 2.34 \\ 37.41 \\ 1.86 \end{pmatrix}, \quad v_2 = \begin{pmatrix} z \\ \frac{D_H}{r_d} \\ \sigma \end{pmatrix} = \begin{pmatrix} 2.34 \\ 8.86 \\ 0.29 \end{pmatrix} \quad (3.63)$$

Instead of D_M we use d_A and as a result:

$$v_1 = \begin{pmatrix} z \\ \frac{d_A}{r_d} \\ \sigma_{d_A} \end{pmatrix} = \begin{pmatrix} z \\ \frac{D_M}{(1+z)r_d} \\ \frac{\sigma}{1+z} \end{pmatrix} = \begin{pmatrix} 2.34 \\ 11.20 \\ 0.56 \end{pmatrix} \quad (3.64)$$

The inverse covariance matrix is:

$$C^{-1} = \begin{pmatrix} (\frac{1}{0.56})^2 & 0 \\ 0 & (\frac{1}{0.29})^2 \end{pmatrix} \quad (3.65)$$

The vector used to define χ^2 is:

$$vec_{Ly-\alpha} = \begin{pmatrix} \frac{d_A}{r_d} - (\frac{d_A}{r_d})_{th} \\ \frac{D_H}{r_d} - (\frac{D_H}{r_d})_{th} \end{pmatrix} \quad (3.66)$$

The χ^2 function is defined as:

$$\chi^2_{Ly-\alpha} = vec_{Ly-\alpha}^T \cdot C^{-1}_{CMB} \cdot vec_{Ly-\alpha} \quad (3.67)$$

Finally the total χ^2 function for the BAO data is the sum of all four individual surveys:

$$\chi^2_{BAO} = \chi^2_{6dFGS, WiggleZ} + \chi^2_{SDSS} + \chi^2_{Ly-\alpha} \quad (3.68)$$

3.10 χ^2 for SNIa data

For the SNIa data we use the Pantheon dataset used in analysis (Alam S. et. al., 2021), which can be found in the GitHub repository (Heavens A., 2009) and the website (Farooq O. & Ratra B., 2013). It consists from 1048 datapoints coming from six different probes. They cover the redshift range $0.01 < z < 2.3$.

The dataset contains the name of each SNIa, its redshift in the CMB and heliocentric frame and its apparent magnitude m_{obs} with its corresponding error $\sigma_{m_{obs}}$. The apparent magnitude m_{obs} is reported after applying color and stretch corrections, as well as corrections from biases of SNIa simulations. (Ross S. M.)

The data vector for the definition of the χ^2 function is:

$$V_i^{pantheon} = m^{obs}(z_i) - m_i^{th} \quad (3.69)$$

where:

$$m_{th} = M + 5 \log_{10}[D_L(z)] + 5 \log_{10} \left(\frac{c/H_0}{1 \text{ Mpc}} \right) + 25 \quad (3.70)$$

D_L is the Hubble - free luminosity distance (Ross S. M.):

$$D_L(z) \equiv \frac{H_0 d_L(z)}{c} \quad (3.71)$$

M and H_0 are degenerate parameters. Thus, they are usually marginalized as nuisance parameters, or use a new parameter, which is a degenerate combination of the two:

$$\begin{aligned}
 \mathcal{M} &= M + 5 \log_{10} \left(\frac{c/H_0}{1 \text{ Mpc}} \right) + 25 \\
 &= M + 5 \log_{10} \left(\frac{299792.458 \text{ km s}^{-1}}{100 h \text{ km s}^{-1} \text{ Mpc}^{-1} 1 \text{ Mpc}} \right) \\
 &= M + 5 \cdot 3.47682 - 5 \log_{10}(h) + 25 \\
 &= M - 5 \log_{10}(h) + 42.38 \quad (3.72)
 \end{aligned}$$

Then,

$$m_{th} = 5 \log_{10}[D_L(z)] + \mathcal{M} \quad (3.73)$$

The χ^2 function is:

$$\chi^2_{Pantheon} = V^i_{Pantheon} \cdot [C_{ij}^{total}]^{-1} \cdot V^i_{Pantheon} \quad (3.74)$$

The total covariance matrix is defined as:

$$C_{ij}^{total} = D_{ij} + C_{syst} \quad (3.75)$$

where

$$D_{ij} = \begin{pmatrix} \sigma_{m.obs,1}^2 & 0 & \dots & 0 \\ 0 & \sigma_{m.obs,2}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \sigma_{m.obs,N}^2 \end{pmatrix} \quad (3.76)$$

and C_{syst} a non-diagonal matrix which is associated with the systematic errors that emerge from bias correction method and can be found in (Alam S. et. al., 2021).

3.11 Total χ^2 function

Like in the BAO analysis, the total χ^2 function is defined as the sum of all the individual functions used in the analysis.

Thus:

$$\chi^2_{total} = \chi^2_{CMB} + \chi^2_{BAO} + \chi^2_{Pantheon} \quad (3.68)$$

The minimized function gives the best - fit parameters for a given cosmological model we want to test.

Chapter 4

Dark Energy

Theorists have tried to explain the origin of the cosmic acceleration (Verde L., 2010) after its discovery. They tried two fundamental ways, either modifying the energy momentum tensor in Einstein equation by introducing ideal fluids with constant equation of state w or scalar field or by modifying gravity.

4.1 Spatially-flat Λ CDM

As we know, the Friedmann equations (1.11), (1.12) are:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

Einstein tried to explain a static universe, which implies that $\dot{a} = 0$. If $k = 1$, the static model is compatible with the first equation, supposing we have the appropriate energy density. But in this case $\ddot{a}$ would never vanish unless P is negative. As a result, Einstein modified his equation and introduced the cosmological constant Λ (Verde L., 2008). The Λ CDM is the most common cosmological model whose energy density is time independent:

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \quad (4.1)$$

Assuming a flat universe with matter and cosmological constant, (1.37) becomes:

$$H^2(z) = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3} \quad (4.2)$$

In terms of dimensionless density parameters, (4.2) becomes:

$$H^2(z) = H_0^2[\Omega_m(1+z)^3 + \Omega_\Lambda] \quad (4.3)$$

Since, we take a flat universe with components only the matter and dark energy, we get:

$$\Omega_m + \Omega_\Lambda = 1 \quad (4.4)$$

This model is the minimal standard model of cosmology, since it has one free parameter which can be constrained through observations. Even though it can't be excluded by the observations, it faces theoretical and observational challenges.

4.1.1 Theoretical challenges

The two fundamental theoretical challenges are the fine - tuning problem (Press W. H. et. al., 2007) and the coincidence problem.

Fine - tuning problem

From the observations of cosmic acceleration and using the unit conversion for km to Mpc and 1/s to eV, we require that:

$$\Lambda \approx H_0^2 = \left(100h \frac{km}{sMpc}\right)^2 \approx (2.132h \times 10^{-42} GeV)^2 \quad (4.5)$$

The corresponding energy density is:

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \approx \frac{(2.132h \times 10^{-42} GeV)^2}{8\pi G} \quad (4.6)$$

After substituting $h \approx 0.7$ and $G \propto m_{pl}^{-2}$ we get:

$$\rho_\Lambda \approx 10^{-47} GeV^4 \quad (4.7)$$

We can assume that the cosmological constant is the vacuum energy that comes from the empty space. The vacuum energy is considered to be the

sum of all zero - point energies of some field with mass m , momentum k and frequency ω up to a cut - off scale $k_{\max} \gg m$ (Press W. H. et. al., 2007]:

$$\rho_{vac} = \int_0^{k_{\max}} \frac{d^3k}{(2\pi)^3} \frac{\sqrt{k^2 + m^2}}{2} = \int_0^{k_{\max}} \frac{4\pi k^2 dk}{(2\pi)^3} \frac{\sqrt{k^2 + m^2}}{2} \quad (4.8)$$

The integral is dominated by large k modes ($k \gg m$). Thus:

$$\rho_{vac} \approx \int_0^{k_{\max}} \frac{4\pi k^2 dk}{(2\pi)^3} \frac{k}{2} = \frac{k_{\max}^4}{16\pi^2} \quad (4.9)$$

We take the Planck scale as the cut - off scale since up to m_{pl} General Relativity is believed to hold. Then:

$$\rho_{vac} \approx \frac{m_{pl}^4}{16\pi^2} \approx \frac{(1.22 \times 10^{19} \text{ GeV})^4}{16\pi^2} \approx 10^{74} \text{ GeV}^4 \quad (4.10)$$

As a result,

$$\frac{\rho_{vac}}{\rho_{\Lambda}} \approx \frac{10^{74} \text{ GeV}^4}{10^{-47} \text{ GeV}^4} \approx 10^{121} \quad (4.11)$$

That means that the predicted value is 121 orders of magnitude larger than the observed one. Even in the super - symmetry, where every boson has a fermion of equal mass as a super - symmetric partner and the vacuum energies of these partners cancel, the cut - off scale is 1 TeV, we have (Liddle A. R., 2015):

$$\rho_{vac} \approx \frac{(10^3 \text{ GeV})^4}{16\pi^2} \approx 6.33 \times 10^9 \text{ GeV}^4 \quad (4.12)$$

In other words, here the predicted value is 56 orders of magnitude larger than the observed one. This is called the fine - tuning or cosmological constant problem.

Coincidence problem

In the Λ CDM nobody can explain why the energies densities of non - relativistic matter and cosmological constant are comparable at the present epoch is the coincidence problem. This is bizarre since (Liddle A. R., 2015):

$$\frac{\rho_{\Lambda}}{\rho_m} \propto a^3 \quad (4.13)$$

which implies that Λ was negligible in the past and will dominate in the future. Considering that Λ is an initial condition in the early Universe, it seems unlikely to have at the present epoch comparable value to matter density while galaxies and other large scale structures have formed. If the energy density of the cosmological constant was a couple orders of magnitude higher in the early Universe, then galaxies and other structures wouldn't have formed, because the accelerating expansion of the Universe would have started earlier. On the other hand, if it was a couple order of magnitude lower, it wouldn't have been detectable. A possible solution to this problem is the anthropic principle. (Liddle A. R., 2015).

Anthropic principle

It was introduced by Carter in 1974. The basic idea is that the world is the way it is, because otherwise there would be no life on Earth and consequently, no one would wonder why the world is the way it is.

According to the weak anthropic principle we are here as one more experimental datum, while the strong version states that the laws of nature are incomplete without the conditions that allow intelligent life to arise. (Liddle A. R., 2007).

4.1.2 Observational challenges

These challenges occur when different observations favor different values of the same parameters and usually these parameters are H_0 and Growth



tension. A more detailed analysis for all tensions and curiosities that concern the Λ CDM model can be found in (Escamilla-Rivera C., 2016).

H_0 tension

The Hubble constant is obtained to have value of $H_0 = 67.4 \pm 0.5 \text{ kms}^{-1}\text{Mpc}^{-1}$, according to data from CMB and BAO measurements, as well as Λ CDM background cosmology. But from local measurements from SNIa data the Hubble constant is obtained to have a value of $H_0 = 74.03 \pm 1.42 \text{ kms}^{-1}\text{Mpc}^{-1}$ (Ross A. J. et al., 2015). This discrepancy is 4.4σ (Anderson L. et. al., 2014) and in general it can range to over 5σ depending on the local data that are used (De Sainte Agathe V. et. al., 2019).

Growth tension

The growth rate and magnitude of linear cosmological perturbations are affected by the matter density parameter Ω_{0m} and the amplitude of the primordial power spectrum, which is defined within the parameter σ_8 , the linear amplitude of matter fluctuations on scale of $8h^{-1}\text{Mpc}$. Observations from Weak Lensing and Redshift Space Distortion, two dynamical probes, indicate that the observed growth rate is weaker than the theoretical prediction in the context of Λ CDM with parameters determined from observational background expansion rate using data from geometrical probes, like SNIa, CMB, BAO.

Dynamical probes favor smaller values of Ω_{0m} , σ_8 , than the geometric ones. This discrepancy varies from around 2σ - 3σ . The tension decreases below 2σ when only the constraints from SNIa data are taken into account or when marginalized confidence contours are used. (Ross S. M.)

4.1.3 Maximum Likelihood Estimation

BAO and CMB Data

Within our Mathematica implementation, we build the χ^2 function for the given model, denoted as $\chi^2(\Omega_{0m}, h)$. For this procedure, we adopt the parameter value $\Omega_b h^2 = 0.02236$. We then perform two distinct statistical analyses: the first relies exclusively on Baryon Acoustic Oscillations (BAO) measurements, while the second incorporates both BAO and Cosmic Microwave Background (CMB) observations (Aghanim N. et. al., 2020)

For the initial approach utilizing only BAO measurements, the expression is:

$$\chi^2(\Omega_{0m}, h) = \chi_{BAO}^2 \quad (4.14)$$

where χ_{BAO}^2 is given by Eq. (3.53).

When combining both the BAO and CMB datasets, the function becomes:

$$\chi^2(\Omega_{0m}, h) = \chi_{BAO}^2 + \chi_{CMB}^2 \quad (4.15)$$

where χ_{CMB}^2 is specified in Eq. (3.36).

The SNIa data are used to constrain the Ω_{0m} parameter using the degenerate combination $\mathcal{M}$. Thus, the function is:

$$\chi^2(\Omega_{0m}, \mathcal{M}) = \chi_{Pantheon}^2 \quad (4.16)$$

Finally, the combined data are described through the function:

$$\chi^2(\Omega_{0m}, \mathcal{M}) = \chi_{Pantheon}^2 + \chi_{BAO}^2 + \chi_{CMB}^2 \quad (4.17)$$

4.2 Spatially-flat w-CDM model

This model is characterized by a spatially - homogeneous fluid with equation of state parameter:

$$w_0 = \frac{p_0}{\rho_0} < -\frac{1}{3} \quad (4.18)$$

Here, w_0 is constant. When it is equal to -1, we have the Λ CDM model. Thus, the Friedmann equation for a flat universe with matter and a spatially - homogeneous fluid with $w < -1/3$ is:

$$H^2(z) = \frac{8\pi G}{3}(\rho_m + \rho_0) \quad (4.19)$$

In terms of dimensionless density parameters, (4.19) takes the following form:

$$H^2(z) = H_0^2[\Omega_m(1+z)^3 + \Omega_{w_0}(1+z)^{3(1+w_0)}] \quad (4.20)$$

with

$$\Omega_m + \Omega_0 = 1 \quad (4.21)$$

For the analysis, the same data from Pantheon, CMB and BAO will be used, while M and h will be examined separately:

$$\chi^2(\Omega_{om}, M, w_{DE}, h) = \chi_{Pantheon}^2 + \chi_{BAO}^2 + \chi_{CMB}^2 \quad (4.22)$$

4.3 Scalar field dark energy models (Quintessence)

For quintessence we take a canonical scalar field minimally coupled, i.e feeling only gravity, passively through the spacetime curvature. A self - interaction is described by the scalar potential $V(\phi)$. Thus, the Lagrangian density is (Scolnic D. M. et. al., 2018):

$$\mathcal{L} = \frac{1}{2}\partial_\mu\partial^\mu\phi - V(\phi) \quad (4.23)$$

The energy momentum is described through Noether prescription as:

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\mu\nu}} \quad (4.24)$$

The tensor reduces to:

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\partial(\sqrt{-g})}{\partial g_{\mu\nu}} \mathcal{L} + \frac{-2}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} \sqrt{-g} \quad (4.25)$$

From the first term we have:



$$\frac{\partial(\sqrt{-g})}{\partial g_{\mu\nu}} = -\frac{1}{2\sqrt{-g}} \frac{\partial(g)}{\partial g_{\mu\nu}} = -\frac{1}{2\sqrt{-g}} \frac{g g^{\alpha\beta} \partial g_{\alpha\beta}}{\partial g_{\mu\nu}} = \frac{1}{2} \sqrt{-g} g_{\mu\nu} \quad (4.26)$$

From the second term we have:

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = \frac{\partial}{\partial g_{\mu\nu}} \left[\frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - V(\varphi) \right] = \frac{1}{2} \delta_\mu^\alpha \delta_\nu^\beta \partial_\alpha \varphi \partial_\beta \varphi = \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi \quad (4.27)$$

As a result, the energy tensor takes the form:

$$T_{\mu\nu} = -g_{\mu\nu} \mathcal{L} - 2 \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \mathcal{L} \quad (4.28)$$

Substituting $\mathcal{L}$ we get:

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \left(\frac{1}{2} \partial_\mu \partial^\mu \varphi - V(\varphi) \right) \quad (4.29)$$

Assuming a scalar field close to spatially uniform, then we can neglect its spatial derivatives compared to its time derivatives. As a result, $T_{\mu\nu}$ is considered to be diagonal.

The energy density and pressure are equal to:

$$\rho_\varphi = T^0_0 = g^{00} T_{00} = \frac{\dot{\varphi}^2}{2} + V(\varphi) \quad (4.30)$$

$$p_\varphi = T^i_i = g^{ii} T_{ii} = \frac{\dot{\varphi}^2}{2} - V(\varphi) \quad (4.31)$$

The equation of state parameter is:

$$w(\varphi) = \frac{\frac{\dot{\varphi}^2}{2} - V(\varphi)}{\frac{\dot{\varphi}^2}{2} + V(\varphi)} = \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)} \quad (4.32)$$

For a field where kinetic energy dominates ($\dot{\varphi}^2 \gg V(\varphi)$):

$$w(\varphi) = \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)} \approx \frac{\dot{\varphi}^2}{\dot{\varphi}^2} \approx 1 \quad (4.33)$$

On the other hand, when potential energy dominates ($\dot{\varphi}^2 \ll V(\varphi)$):

$$w(\varphi) = \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)} \approx \frac{-2V(\varphi)}{2V(\varphi)} \approx -1 \quad (4.34)$$

For $\dot{\varphi} = 0$ we get $w = -1$. To play the role of dark energy, quintessence should have (Scolnic D. M. et. al., 2018):

$$\begin{aligned} w(\varphi) < -\frac{1}{3} &\Rightarrow \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)} < -\frac{1}{3} \Rightarrow 3\dot{\varphi}^2 - 6V(\varphi) < -\dot{\varphi}^2 - 2V(\varphi) \\ &\Rightarrow \dot{\varphi}^2 < V(\varphi) \end{aligned} \quad (4.35)$$

However, this is not enough, since dark energy domination today requires $w \simeq -1$ not only for today, but for an extended period of time (roughly, between $z \simeq 1$ until now). Thus, we need not only (4.35) to be valid, but the inequality should also be valid for their time derivatives:

$$\begin{aligned} \left| \frac{d}{dt} \dot{\varphi}^2 \right| < \left| \frac{d}{dt} V(\varphi) \right| &\Rightarrow |2\ddot{\varphi}\dot{\varphi}| < \left| \frac{dV(\varphi)}{d\varphi} \dot{\varphi} \right| \\ \Rightarrow |\ddot{\varphi}| < \left| \frac{V'(\varphi)}{2} \right| &< |V'(\varphi)| \end{aligned} \quad (4.36)$$

where $V'(\varphi) = dV(\varphi)/d\varphi$.

(4.35), (4.36) are called Slow - Roll condition and can become negative if the scalar field varies slowly with time.

In general, the dynamics of a homogeneous scalar field is determined by the Klein - Gordon equation:

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0 \quad (4.37)$$

Assuming a flat universe with matter and quintessence, the Friedmann equation takes the form:

$$H^2(z) = \frac{8\pi G}{3}(\rho_m + \rho_\varphi) = \frac{8\pi G}{3}\left(\rho_m + \frac{\dot{\varphi}^2}{2} + V(\varphi)\right) \quad (4.38)$$

In terms of dimensionless density parameters:

$$H^2(z) = H_0^2 \left[\Omega_{0m}(1+z)^3 + \Omega_{0\varphi} \exp\left(\int_0^z \frac{3[1+w(z')]}{1+z'} dz'\right) \right] \quad (4.39)$$

where (1.27) is used to define ρ_φ . Also,

$$\Omega_{0m} + \Omega_{0\varphi} = 1 \quad (4.40)$$

In this model the parameters are Ω_{0m} and the parameters used to parameterize $w(z)$.

4.3.1 Chevalier - Polanski - Linder (CPL) model

The model was a proposal from Chevalier - Polanski and Linder, according to which (Scolnic D. M. et. al., 2018), (Kazantzidis L. & Perivolaropoulos L., 2020):

$$w(a) = w_0 + w_a(1-a) = w_0 + w_a\left(1 - \frac{1}{1+z}\right) \quad (4.41)$$

which is a Taylor expansion around $a = 1$, keeping only the linear terms:

$$w(a) = w(a=1) + (a-1)w'(a=1) + O(a^2) \quad (4.42)$$

The present value is:

$$w(a_0) = w_0 + w_a(1-a_0) = w_0 \quad (4.43)$$

The slope is:

$$w'(a) = \frac{dw(a)}{da} = -w_a \quad (4.44)$$

The CPL model is the third model that will be examined in this paper.

4.4 Phantom energy model

It is worth to mention phantom energy models, they are characterized by constant equation of state $w < -1$. Thus, the acceleration of the Universe is even bigger, something that leads to the so- called Big Rip, in which all structures will be destroyed and dark energy will dominate even more. The phantom energy can emerge in many ways, like a negative kinetic term or modification of the Friedmann equations. In (Riess A. G., 1998), a toy model was made. Phantom energy seems to work in the expansion of universe, but the particles of this energy decay very fast to gravitons until a cut - off scale of 100 MeV. For this reason, the existence of phantom fields is difficult, but it cannot completely be excluded as an idea.

4.5 Calculation of χ^2 and contours.

Λ CDM model

The codes that I ran in Mathematica was about the calculation of χ^2 and contours about the parameters on which the function depends on. At first, the calculations were only about the Λ CDM model and were either from BAO and CMB measurements separately. Then, BAO and CMB data were combined. After that, Pantheon data were added, in order to see how much the parameters are constrained. The parameters were the density parameter Ω_{om} , the degenerate magnitude $\mathcal{M}$ and h . The results were the following figures and the table:

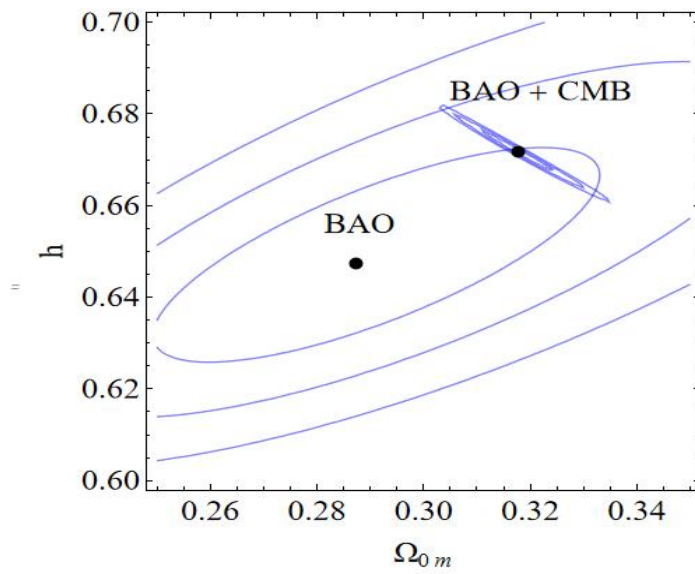


Figure 4.1: The 1σ - 3σ confidence contours in the parametric space (h, Ω_{0m}) due to the BAO and the BAO + CMB data

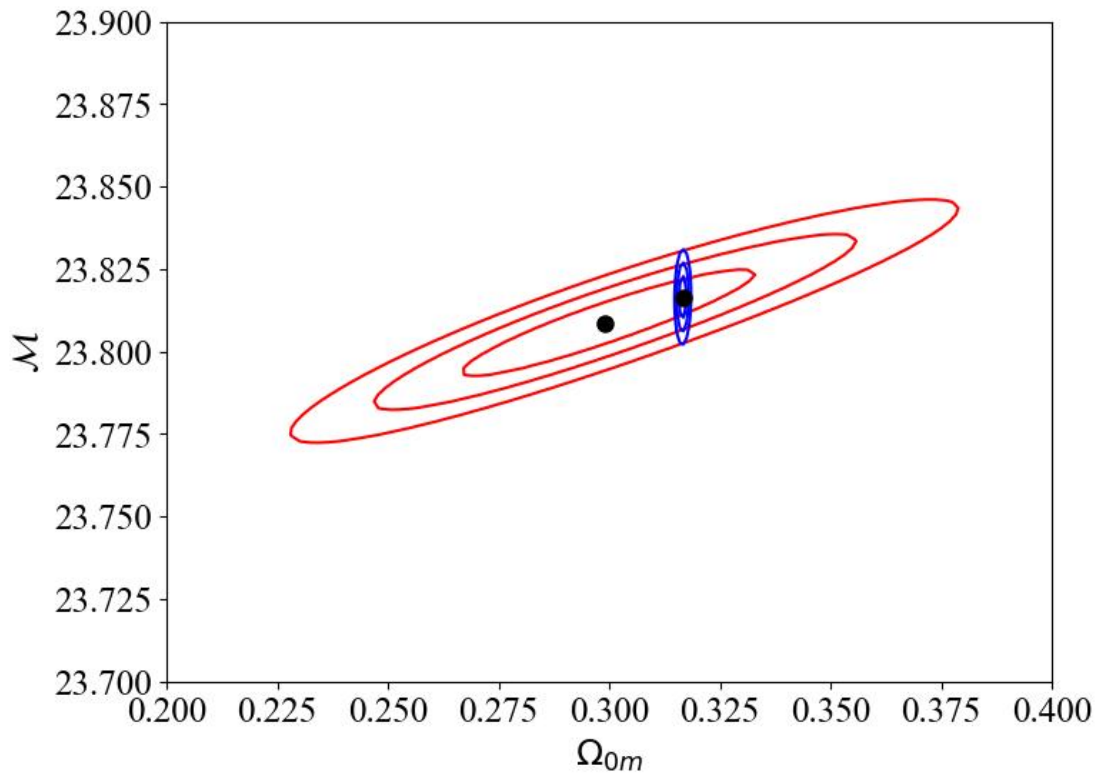


Figure 4.2: The 1σ - 3σ confidence contours in the parametric space $(\mathcal{M}, \Omega_{0m})$. The red contour corresponds only to the full Pantheon dataset while the blue contour corresponds to the Pantheon+CMB+BAO data combination.

	BAO +CMB	SNla	Combined
Ω_{0m}	0.3175 ± 0.0059	0.299 ± 0.022	0.3166 ± 0.0057
$\mathcal{M}$	-	23.809 ± 0.011	23.817 ± 0.049
h	0.6720 ± 0.0039	-	0.6725 ± 0.0038
χ^2	6.531	1025.63	1032.79

Table 4.1: The best fit - parameters of the Λ CDM model based on different measurements

We can see that in both figures the parameters are constrained when the CMB data are mainly introduced to the calculations. Especially in fig.2, the CMB data constrain a lot the parameter of Ω_{0m} , something that can be seen through the elongated ellipse in its x-direction.

wCDM - CPL model

Afterwards, we make the same thing for the wCDM model where w_0 is a parameter and the CPL model, where w_0 and w_a are parameters.

	Λ CDM	wCDM	CPL
Ω_{0m}	0.3175 ± 0.0059	0.3144 ± 0.022	0.3145 ± 0.0057
w_0	-1	-1.01 ± 0.03	-1.08 ± 0.15
w_a	-	-	0.24 ± 0.47
$\mathcal{M}$	23.817 ± 0.049	—	—
M		-19.42 ± 0.02	-19.43 ± 0.02
h	0.6725 ± 0.0038	0.675 ± 0.008	0.674 ± 0.011
χ^2	1032.79	1032.69	1031.82
AIC	1038.79	1040.69	1041.82
BIC	1046.72	1053.59	1059.68

Table 4.2: The best fit - parameters for the three models based on measurements from BAO, CMB, Pantheon data

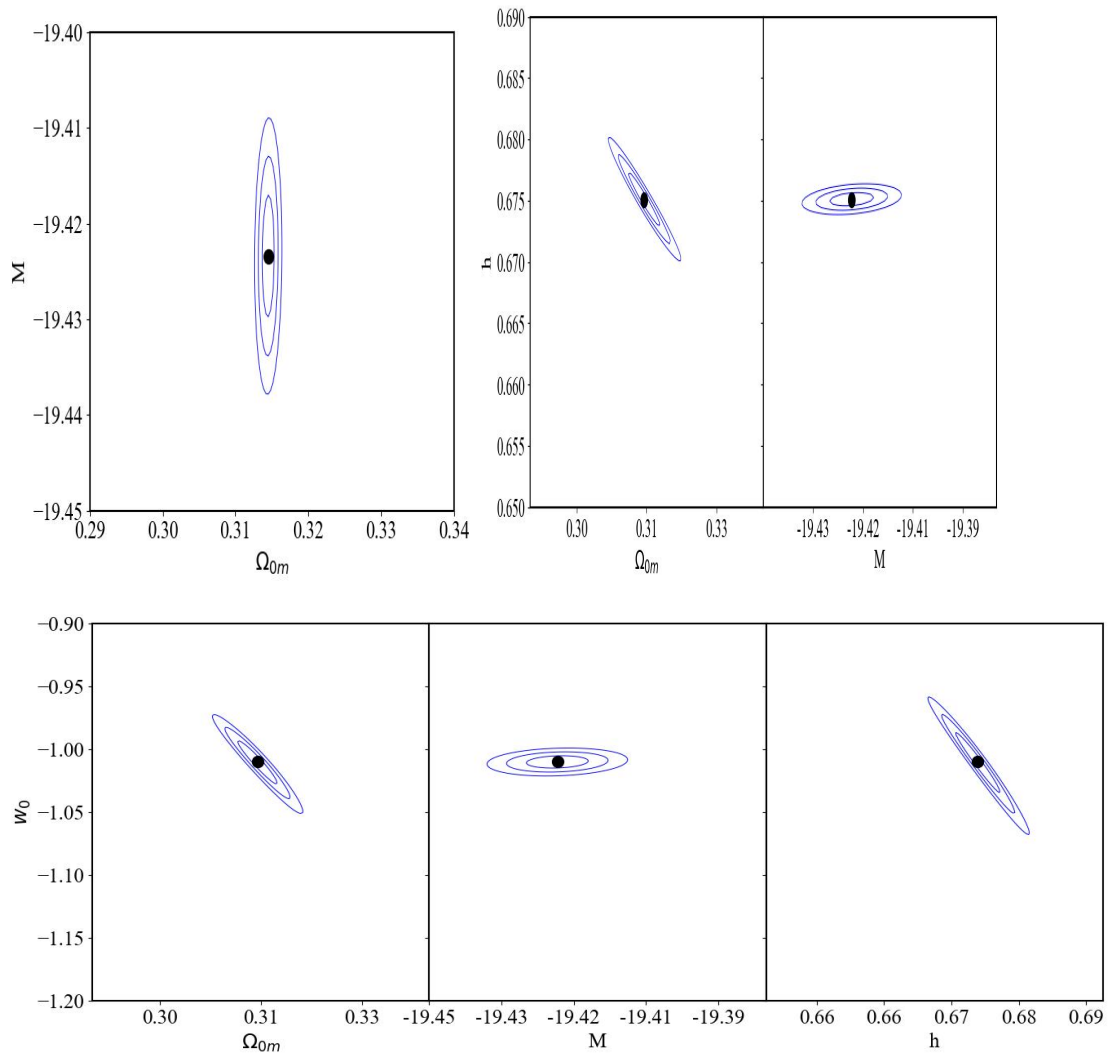


Figure 4.3: Contours in the parametric spaces of all combinations of w CDM parameters

The w CDM and CPL model give about the same value for Ω_{0m} , while CPL gives even smaller value for w_0 than the other two models.

Riess point

We also add the Riess point (Riess G.A et al, 2019), so we have:

$$\chi^2 = \chi_{Pantheon}^2 + \chi_{BAO}^2 + \chi_{CMB}^2 + \chi_{Riess}^2 \quad (4.45)$$

with

$$\chi_{Riess}^2 = \frac{(h - h_{riess})^2}{\sigma_{Riess}^2} = \frac{(h - 0.7403)^2}{0.0142^2} \quad (4.46)$$

	Λ CDM	wCDM	CPL
Ω_{0m}	0.3118 ± 0.0055	0.298 ± 0.007	0.298 ± 0.013
w_0	-1	-1.07 ± 0.03	-1.12 ± 0.16
w_a	-	-	0.24 ± 0.47
$\mathcal{M}$	23.814 ± 0.049	—	—
M		-19.39 ± 0.02	-19.43 ± 0.02
h	0.6758 ± 0.0038	0.693 ± 0.008	0.674 ± 0.011
χ^2	1054.49	1047.87	1047.34

Table 4.3: The best fit - parameters for the three models based on measurements from BAO, CMB, Pantheon data and Riess point

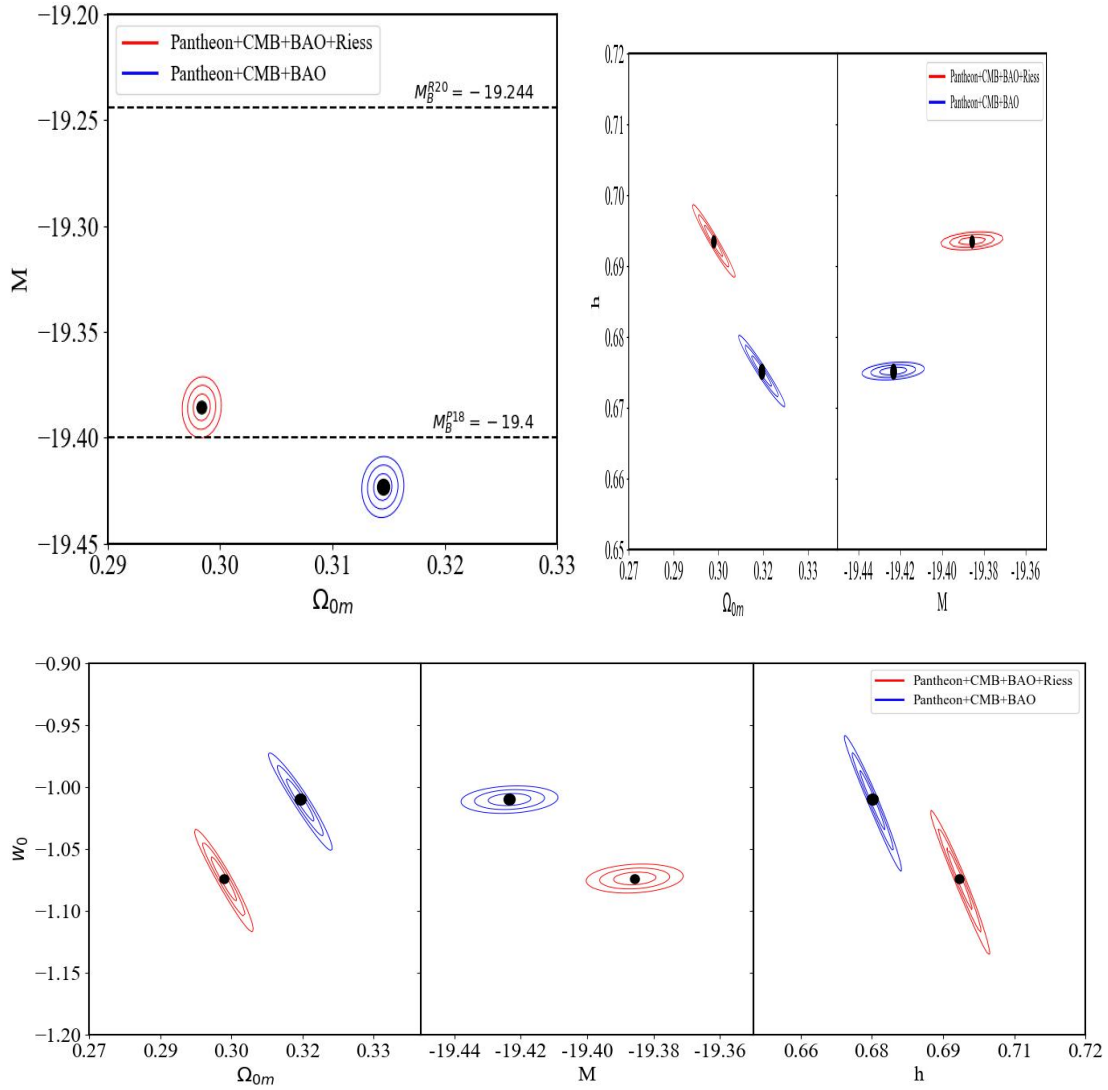


Figure 4.4: The parametric space for the w CDM model with and without Riess point

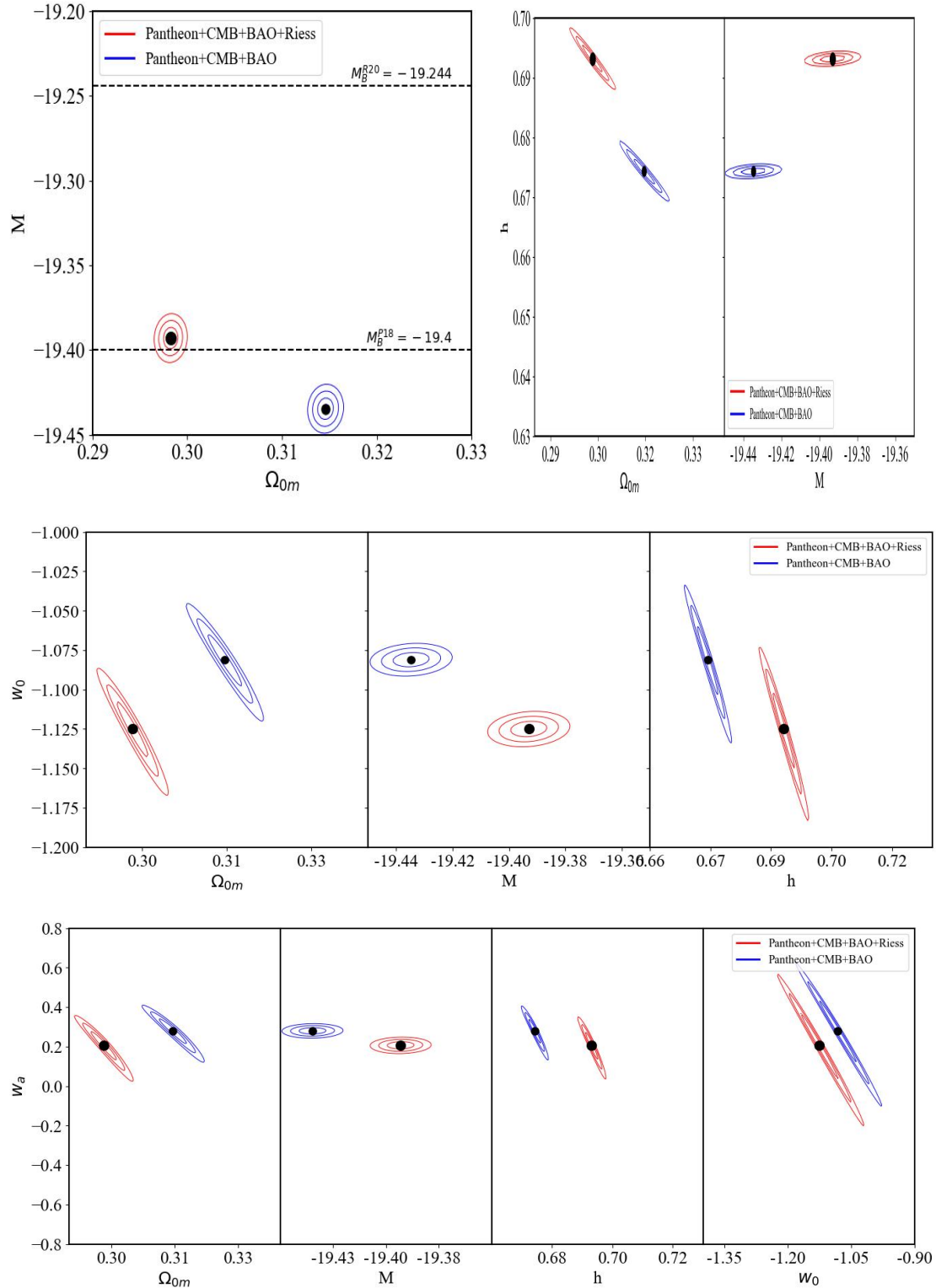


Figure 4.5: The parametric space for all parameters of CPL model with and without Riess point

The addition of Riess point has the same effects in both models, higher value for the Hubble constant, lower values for the matter density and equation of state. The addition of the Riess point also relieves the tension between the absolute magnitude calculated and M_B^{P18} , still remains in tension with M_B^{R20} .

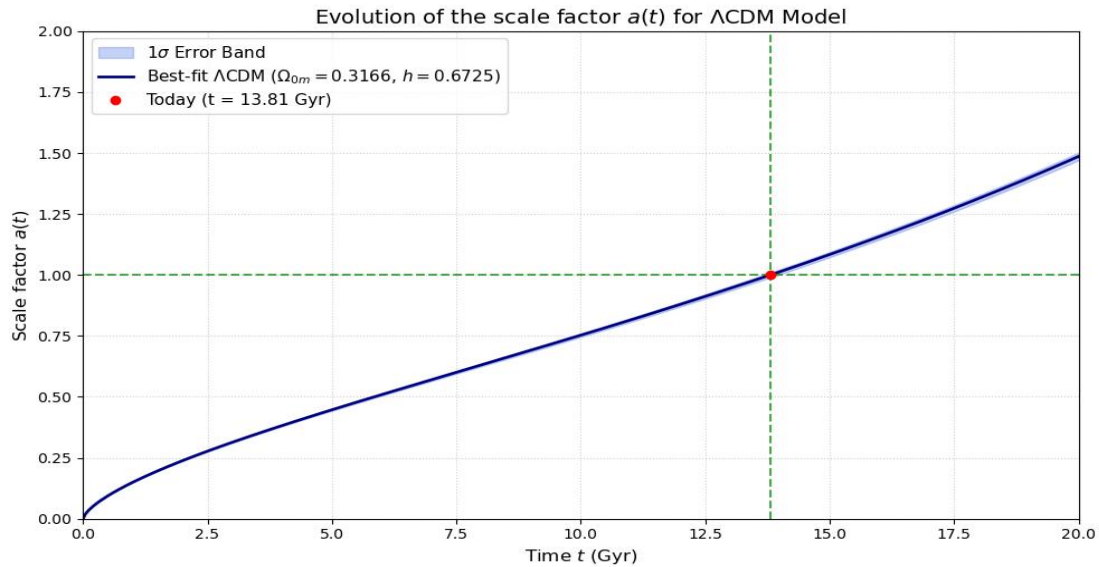


Figure 4.6: The evolution of scale factor and its error band

Final results

Since Λ CDM has lower AIC and BIC values than the other two models, and since the value of equation of state for the Λ CDM model is not excluded from the values of best-fit parameters of the other two models, the Λ CDM model is preferred over the other two. So, using a Python code, the Friedmann equations, and the best - fit parameters of the model analysis, the evolution of the scale factor is in fig. 4.6. The age of the universe is also calculated to 13.8 billion years.



Conclusion

In this thesis, the purpose was to find the best fit for the scale factor $a(t)$. At first, the theory of the cosmological models and the distances in the universe was developed. We used data analysis techniques and especially maximum likelihood estimation to constrain a number of spatially - flat cosmological models. In order to do that, we construct the χ^2 function for each model using observational data from Type Ia Supernovae observations coming from the Pantheon dataset as well as Baryonic Acoustic Oscillations (BAO) and Cosmic Microwave Background (CMB) measurements coming from recent surveys. The aim is to minimize it to obtain the best-fit values for the parameters. We also want to make a comparison for all possible cosmological models. During the analysis, we also use the Riess point.

The CMB data can constrain better the spatially - flat cosmological models in the Λ CDM model, while SN data prefer smaller values for matter density and higher ones for Hubble constant. When it came up to the comparison of the three models, wCDM and CPL models prefer lower values for matter density and higher ones for Hubble constant than Λ CDM.

Although, Λ CDM has bigger value for χ^2 , it has lower AIC and BIC values than the other two models, so it is preferred over them, and the final result is the evolution of the scale factor $a(t)$ using the best - fit parameters of the Λ CDM model.

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